OSCILLATORY PROPERTIES OF LINEAR THIRD-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Separation theorems, distribution of zeros of solutions, and disconjugacy criteria for linear third-order differential equations are discussed. For instance, it is proved that the equation $y'''+py''+qy'+ry=0$, where $p\in C''$, $q\in C'$, and $r\in C$ on an interval I, is disconjugate on I if p does not change sign and if $q\leqq 0$, $r\geqq 0$, $q-2p'\leqq 0$, and $r-q'+p''\leqq 0$ on I.

We shall consider the third-order equation of the form

$$y''' + py'' + qy' + ry = 0, \tag{1}$$

where p, q, and r are real-valued, continuous functions defined on an interval I. The oscillatory behavior of solutions of (1) has been extensively studied by Hanan [4], Azbelev and Caljuk [1], Lazer [6], and Barrett [2]. An up-to-date review of the oscillation theory of the linear third-order differential equation may be found in a recent article by Barrett [3].

In this paper we shall discuss separation theorems, distribution of zeros of solutions, and disconjugacy criteria for equation (1). (1) is said to be *disconjugate* on the interval I if no nontrivial solution of (1) has more than two zeros (counting multiplicities) on I. A nontrivial solution of (1) is said to be *oscillatory* on I if it has an infinite number of zeros on I. If (1) has an oscillatory solution, it is said to be *oscillatory*.

THEOREM 1. *Assume that* $p, q, r\in C[a, b]$ *and that* $p\leqq 0$, $q\leqq 0$, $r\geqq 0$ *on* $[a, b]$. *Let* u *and* v *be two solutions of* (1), *and put*

$$D_{ij} = \begin{vmatrix} u^{(i)} & v^{(i)} \\ u^{(j)} & v^{(j)} \end{vmatrix}, \qquad 0\leqq i<j\leqq 2, \quad i, j = 0, 1, 2.$$

If $D_{01}(a)$, $D_{02}(a)$, *and* $D_{12}(a)$ *are nonnegative, but not all zero, then*

(i) *neither* u *nor* v *have a double zero on* $(a, b]$,

(ii) $u^{(k)}$ *and* $v^{(k)}$ *cannot have a common zero on* $(a, b]$, $k=0, 1$; *and*

(iii) *between any two zeros of* u *on* $(a, b]$, *there is a zero of* v.

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This theorem is an immediate consequence of the following theorem.

THEOREM 2. *Let D_{ij} be defined as in Theorem* 1. *Under the hypotheses in Theorem* 1, *we have* $D_{01}(x) > 0$, $x \in (a, b]$.

PROOF. We use a method due to Mikusiński [7]. It is easily confirmed that

$$D_{01}' = D_{02},$$
$$D_{02}' = D_{12} - pD_{02} - qD_{01}, \tag{2}$$
$$D_{12}' = -pD_{12} + rD_{01}.$$

We may regard the above relations as a system of first-order differential equations satisfied by D_{01}, D_{02}, and D_{12}. Since the coefficients of system (2) and the initial values $D_{01}(a)$, $D_{02}(a)$, and $D_{12}(a)$ are all nonnegative by hypothesis, the solutions D_{01}, D_{02}, and D_{12} are all nonnegative throughout the interval $[a, b]$. This being the case, a glance at system (2) shows that $D_{ij}'(x) \geqq 0$, $x \in (a, b]$, i.e., D_{ij} is nondecreasing on $(a, b]$. By assumption, $D_{kl}(a) > 0$ for some k and l, $0 \leqq k < l \leqq 2$. If $D_{01}(a) > 0$, then $D_{01}(x) > 0$, $x \in (a, b]$, as asserted. If $D_{02}(a) > 0$, then $D_{02}(x) > 0$ for $x \in [a, b]$; and it follows from $D_{01}' = D_{02}$ that D_{01} is positive on $(a, b]$. Finally, if $D_{12}(a) > 0$, then $D_{12}(x) > 0$, $x \in [a, b]$. Therefore, the positivity of D_{02} on $(a, b]$ follows from the second equation in (2), and again we conclude from $D_{01}' = D_{02}$ that D_{01} is positive on $(a, b]$. This completes the proof.

PROOF OF THEOREM 1. (i) and (ii) are immediate consequences of the inequality $D_{01}(x) > 0$, $x \in (a, b]$. To prove (iii), assume the contrary: let α and β be two consecutive zeros of u on $(a, b]$ and assume that v does not vanish on (α, β). Then the function $w \equiv u/v$ is continuously differentiable on $[\alpha, \beta]$ and $w(\alpha) = w(\beta) = 0$, since $v(\alpha) \neq 0$ and $v(\beta) \neq 0$ by (ii). Thus, by Rolle's theorem, there exists a point $\xi \in (\alpha, \beta)$, $a < \alpha < \beta \leqq b$, such that

$$0 = w'(\xi) = \frac{u'(\xi)v(\xi) - u(\xi)v'(\xi)}{v^2(\xi)} = \frac{-D_{01}(\xi)}{v^2(\xi)},$$

i.e., $D_{01}(\xi) = 0$, contrary to the inequality $D_{01}(x) > 0$, $x \in (a, b]$. This contradiction proves (iii).

REMARKS. 1. In the proof of Theorem 2, we established that $D_{01}(x) \geqq 0$, $D_{02}(x) \geqq 0$, and $D_{12}(x) \geqq 0$, $x \in (a, b]$. If we assume that there exists an $\epsilon > 0$ such that $r(x) > 0$, $x \in (a, a+\epsilon)$, the inequalities

($\geqq$) may be strengthened to strict inequalities ($>$). These strict inequalities yield results similar to those in Theorem 1. For instance, the inequality $D_{12}(x)>0$, $x\in(a, b]$, implies that

(i) neither u' nor v' have a double zero on $(a, b]$, and

(ii) $u^{(k)}$ and $v^{(k)}$ cannot have a common zero on $(a, b]$, $k=1, 2$.

2. If we assume in Theorem 2 that $D_{01}(a)\geqq 0$, $D_{02}(a)>0$, and $D_{12}(a)\geqq 0$, it is easily confirmed that $D_{01}(x)>0$, $D_{02}(x)>0$, and $D_{12}(x)\geqq 0$, $x\in(a, b]$. Similarly, the inequalities $D_{01}(a)\geqq 0$, $D_{02}(a)\geqq 0$, and $D_{12}(a)>0$ yield $D_{01}>0$, $D_{02}>0$, and $D_{12}>0$ on $(a, b]$.

COROLLARY 1. *Let u be a solution of* (1) *satisfying the initial conditions*

$$u(\beta) = u'(\beta) = 0, \quad u''(\beta) > 0, \qquad \beta \in I.$$

If $p\leqq 0$, $q\leqq 0$, and $r\geqq 0$ on the interval I, then

$$u(x) > 0, \quad u'(x) < 0, \quad u''(x) > 0, \qquad x \in I, x < \beta. \tag{3}$$

PROOF. Evidently, there exists an $\epsilon>0$ such that the inequalities in (3) hold on the interval $(\beta-\epsilon, \beta)$. If an inequality in (3) fails to hold, there would exist a first point α, to the left of β, at which $u(\alpha)>0$, $u'(\alpha)<0$, and $u''(\alpha)=0$. Let v be the solution of (1) satisfying the initial conditions $v(\alpha)=0$, $v'(\alpha)=u(\alpha)$, and $v''(\alpha)=0$. Then $D_{01}(\alpha) =u^2(\alpha)>0$ and $D_{02}(\alpha)=D_{12}(\alpha)=0$. Hence, by part (i) of Theorem 1, the solution u cannot have a double zero at $x=\beta$, contrary to our hypothesis. This contradiction proves (3).

We next require an elementary generalization of a result of Lazer [6, Lemma 1.1]. We state it as a lemma for convenient reference.

LEMMA 1. *If $p\geqq 0$, $q\leqq 0$, $r\geqq 0$ on I, and if y is a solution of* (1) *satisfying the initial conditions*

$$y(\beta) \geqq 0, \quad y'(\beta) \leqq 0, \quad y''(\beta) > 0, \qquad \beta \in I, \tag{4}$$

then

$$y(x) > 0, \quad y'(x) < 0, \quad y''(x) > 0, \qquad x \in I, x < \beta. \tag{5}$$

PROOF. It follows from (4) that there exists an $\epsilon>0$ such that the inequalities in (5) hold on the interval $(\beta-\epsilon, \beta)$. If the inequalities in (5) do not hold for some point $x_0\in I$, $x_0<\beta$, then there would exist a first point $\alpha\in I$, $\alpha<\beta$, with $y''(\alpha)=0$. However, $y'''(x)\leqq 0$ on (α, β) which is a contradiction.

From Corollary 1 and Lemma 1, we obtain the following theorem.

THEOREM 3. *If p does not change sign, and if $q\leqq 0$ and $r\geqq 0$ on the* *interval I, the only solution of* (1) *satisfying the boundary conditions $y(\alpha)=y(\beta)=y'(\beta)=0$, $\alpha, \beta \in I$, $\alpha<\beta$, is the trivial one.*

REMARK. In the above theorem we may relax somewhat the condition that p does not change sign on I. The p may change its sign on I provided that it changes only once from a positive to a negative sign, i.e., for some $\xi \in I$, $p(x) \geqq 0$, $x \leqq \xi$, $x \in I$, and $p(x) \leqq 0$, $x \geqq \xi$, $x \in I$.

As we shall see presently, Theorem 3 gives rise to a disconjugacy condition for (1). This condition is somewhat different from other disconjugacy conditions known to the author: the length of interval of disconjugacy does not appear explicitly in the condition. To obtain the sufficient condition, we use the following well-known results [1]: If (1) has a nontrivial solution with three zeros on I, then there is a nontrivial solution y of (1) which satisfies at least one set of the boundary conditions

$$y(a) = y'(a) = y(b) = 0, \tag{6}$$

$$y(a) = y(b) = y'(b) = 0, \tag{7}$$

where $a, b \in I$ and $a<b<\infty$. Furthermore, if (1) with $p \in C''$, $q \in C'$ and $r \in C$ has a nontrivial solution satisfying condition (6) $\{(7)\}$, then its adjoint equation

$$y''' - py'' + (q - 2p')y' - (r - q' + p'')y = 0 \tag{8}$$

has a nontrivial solution which satisfies condition (7) $\{(6)\}$.

In view of these results, we see that (1) is disconjugate on I if and only if neither (1) nor its adjoint equation (8) have a nontrivial solution satisfying boundary condition (7). According to Theorem 3, the adjoint equation has no nontrivial solution satisfying (7), provided that p does not change sign on I and that $q-2p' \leqq 0$ and $r-q'+p'' \leqq 0$ on I. This proves the following theorem.

THEOREM 4. *Assume that $p \in C''$, $q \in C'$, and $r \in C$ on an interval I. If p does not change sign, and if $q \leqq 0$, $r \geqq 0$, $q-2p' \leqq 0$, and $r-q'+p'' \leqq 0$ on I, then* (1) *is disconjugate on I.*

Our next results are concerned with the case $p \leqq 0$, $q \leqq 0$, and $r \leqq 0$. The following lemma generalizes a result of Lazer [6, Lemma 2.1]. We shall state it for the nth-order differential equation.

LEMMA 2. *Let p_i, $i=0, 1, \cdots, n-1$, be continuous on an interval $[a, \infty)$ and let y be a solution of the equation*

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = 0, \tag{9}$$

satisfying the initial conditions

$$y^{(i)}(\alpha) \geqq 0, \qquad i = 0, \cdots, k-1, k+1, \cdots, n-1,$$
$$y^{(k)}(\alpha) > 0, \qquad \alpha \in [a, \infty).$$

If $p_i \leqq 0$ *on* $[a, \infty)$, $i = 0, 1, \cdots, n-1$, *then*

$$y^{(i)}(x) > 0, \qquad x > \alpha, \quad i = 0, 1, \cdots, k,$$
$$y^{(i)}(x) \geqq 0, \qquad x \geqq \alpha, \quad i = k+1, \cdots, n,$$

and

$$\lim_{x \to +\infty} y^{(i)}(x) = +\infty, \qquad i = 0, 1, \cdots, k-1.$$

PROOF. Put $u_i = y^{(i)}$, $i = 0, 1, \cdots, n$. Then (9) yields the differential system

$$\begin{aligned} u_0' &= u_1 \\ u_1' &= u_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{n-1}' &= -p_{n-1}u_{n-1} - \cdots - p_0 u_0. \end{aligned} \tag{10}$$

All the coefficients of system (10) are nonnegative and the initial values $u_i(\alpha)$, $i = 0, 1, \cdots, n-1$, are nonnegative. Hence, $u_i(x) \geqq 0$, $x \in [\alpha, \infty)$, and consequently we conclude that $u_i(x)$ is nondecreasing on $[\alpha, \infty)$, $i = 0, 1, \cdots, n$. In particular, $u_k(x) \geqq u_k(\alpha) > 0$, $x \in [\alpha, \infty)$. The strict positivity of u_i on (α, ∞), $i = 0, 1, \cdots, k-1$, now follows successively from $u_i' = u_{i+1}$, $i = 0, 1, \cdots, k-1$, and this in turn yields

$$\lim_{x \to +\infty} y^{(i)}(x) = y^{(i)}(\alpha) + \lim_{x \to +\infty} \int_\alpha^x y^{(i+1)}(\xi) d\xi = +\infty,$$

$i = 0, 1, \cdots, k-1$.

THEOREM 5. *Assume that* $p \leqq 0$, $q \leqq 0$, *and* $r \leqq 0$ *on an interval* $[a, \infty)$. *Then the zeros of two bounded linearly independent solutions of* (1) *separate. If* (1) *has a bounded oscillatory solution* u, *then the zeros of any other oscillatory solution separate the zeros of* u.

PROOF. Let y_1 and y_2 be two bounded linearly independent solutions. If the zeros of y_1 and y_2 do not separate, we can find a constant C such that $y \equiv y_1 + Cy_2$ has a double zero at some point $\xi \in [a, \infty)$, as in the proof of (iii) of Theorem 1. Thus, $y(\xi) = y'(\xi) = 0$; moreover, we may assume $y''(\xi) > 0$ (if $y''(\xi) < 0$, take $-y$). Therefore, by Lemma 2,

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} [y_1(x) + Cy_2(x)] = +\infty,$$

contrary to the boundedness of y_1 and y_2. This contradiction proves that the zeros of y_1 and y_2 separate.

The second assertion may be proved in a similar way.

THEOREM 6. *Assume that* $p \leqq 0$, $q \leqq 0$, *and* $r \leqq 0$ *on an interval* $[a, \infty)$. *If* (1) *is oscillatory on* $[a, \infty)$, *then, for any nonnegative integer* N, *there exists a nontrivial solution of* (1) *which has exactly* N *simple zeros on* $[a, \infty)$.

PROOF. Lemma 2 implies that every zero on $[a, \infty)$ of an oscillatory solution is simple. Let $x_k \geqq a$, $k=1, 2, \cdots$, be the kth zero on $[a, \infty)$, to the right of a, of an oscillatory solution y. In view of Lemma 2, it is clear that the assertion holds for $N=0, 1, 2$, whether or not (1) is oscillatory. In order to prove the theorem for $N=3, 4, \cdots$, we shall use a continuity argument (cf. [5, Lemma 2.2]). Let y_1, y_2, and y_3 be linearly independent solutions of (1), and define

$$w(x; x_1, x_N) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1(x_1) & y_2(x_1) & y_3(x_1) \\ y_1(x_N) & y_2(x_N) & y_3(x_N) \end{vmatrix}.$$

Evidently, w is a nontrivial solution of (1) and vanishes at x_k, $k=1, 2, \cdots$. Furthermore, w is a continuous function of x_1 and x_N, and w remains a solution as x_1 and x_N range over the interval $[a, \infty)$. We note that $y=Kw$ for some constant K. For, if y and w were linearly independent, there would exist a nontrivial solution u such that $u(x_1)=u'(x_1)=u(x_N)=0$, contrary to Lemma 2. Fix x_1 and let x_N approach x_1, and observe the movements of zeros x_2, $x_3, \cdots, x_{N-1}$. Since the N $(\geqq 3)$ zeros cannot coalesce at x_1, the intervening zeros $x_2, x_3, \cdots, x_{N-1}$ must disappear from the interval $[x_1, x_N]$, as x_N approaches x_1. The intervening zeros can disappear only in pairs or across the boundary point x_1 or x_N. We assert that the zeros cannot disappear in pairs. If two zeros x_l and x_{l+1}, $2 \leqq l \leqq N-2$, are to disappear in pair, they must first coalesce to a double zero ζ $(=x_l=x_{l+1})$, as the zero x_N approaches x_1. But this is impossible because (1) cannot have a nontrivial solution with a double zero at ζ and a zero x_N, $\zeta < x_N$. For the same reason, the intervening zeros cannot disappear across the point x_1. Therefore, the zeros must disappear across x_N. This proves that there exists a solution u which has exactly $N-2$ simple zeros on $[x_1, x_N)$ and a double zero at x_N. Without loss of generality, we may assume that $x_1=a$ because if $x_1>a$ we can move the double zero x_N toward a until x_1 coincides with a. Moreover, we see that u cannot vanish on (x_N, ∞). Since we may assume $u''(x_N)>0$, we have $u(x)>0$ and $u'(x)>0$ for $x>x_N$ by Lemma 2. The double zero of u at x_N can be separated into two simple zeros, without losing or gaining any other zeros on $[a, \infty)$. This may be accomplished by holding the zero x_1 at a and moving one zero at x_N toward a until the double zero at x_N separates into two simple zercs. This establishes the existence of a nontrivial solution which has exactly N simple zeros on $[a, \infty)$.

In our last theorem, we prove a sufficient condition for disconjugacy of (1).

THEOREM 7. *Let $p, q\in C'(a, b)$ and $r\in C(a, b)$. If $p\leqq 0$, $q\leqq 0$, $r\leqq 0$, $p^2+p'+q\leqq 0$, and $pq-r+q'\leqq 0$ on (a, b), then* (1) *is disconjugate on (a, b).*

PROOF. It suffices to prove that (1) is disconjugate in every subinterval (α, β), $a<\alpha<\beta<b$. Let u and v be the solutions of (1) satisfying the initial conditions

$$u(\alpha) = 1, \quad u'(\alpha) = 0, \quad u''(\alpha) = 0,$$
$$v(\alpha) = 0, \quad v'(\alpha) = 1, \quad v''(\alpha) = 0,$$

respectively. Define D_{ij} as in Theorem 1. Due to a result of Pólya [8, Theorem II], (1) would be disconjugate on (α, β) if we could show that $u>0$ and $D_{01}>0$ throughout the interval (α, β). Since $p\leqq 0$, $q\leqq 0$, and $r\leqq 0$ on (α, β), it follows from Lemma 2 that $u>0$ on (α, β). To prove $D_{01}>0$, we put $D=D_{12}-pD_{02}-qD_{01}$. Then D_{01}, D_{02}, and D satisfy the differential system

$$D_{01}' = D_{02}, \quad D_{02}' = D,$$
$$D' = -(pq - r + q')D_{01} - (p^2 + p' + q)D_{02} - 2pD,$$

where the coefficients are all nonnegative, and the initial conditions

$$D_{01}(\alpha) = 1, \quad D_{02}(\alpha) = 0,$$
$$D(\alpha) = -q(\alpha) \geqq 0.$$

Thus, $D_{01}>0$ on (α, β), as in the proof of Theorem 2. This completes the proof.

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