EXISTENCE OF UNIVERSAL MEMBERS IN CERTAIN FAMILIES OF BASES OF BANACH SPACES¹

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ABSTRACT. In a recent paper A. Pełczyński proved the existence of universal Schauder bases for several important families of bases. In the present paper some new existence problems are settled. For example, it is proved that the family of boundedly complete bases does not have a universal member.

1. The purpose of this note is to introduce a simple tool which, together with a deep result of A. Pełczyński [3], enables us to settle problems concerning the existence of universal members in several families of bases (see the definitions below). We then use similar methods to verify a conjecture of Pełczyński. We begin with a few definitions. A sequence $\{x_n\}$ of elements of a Banach space X is called a *basis* if each $x \in X$ has a unique representation $x = \sum_{i=1}^{\infty} t_i x_i$ where $\{t_i\}$ are scalars. A basis $\{x_n\}$ is called *seminormalized* [*normalized*] if $0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| < \infty$ [$\|x_n\| = 1$ for all n], it is called *monotone* if for any $n > 1$ and scalars $t_1, t_2, \dots, t_n$ $\|\sum_{i=1}^{n-1} t_i x_i\| \leq \|\sum_{i=1}^n t_i x_i\|$. Let $\{x_n\}$ and $\{y_n\}$ be bases of the Banach spaces X and Y respectively. Then $\{x_n\}$ is said to be equivalent to $\{y_n\}$ provided that for every sequence $\{t_i\}$ of scalars $\sum_{i=1}^{\infty} t_i x_i$ converges if and only if $\sum_{i=1}^{\infty} t_i y_i$ converges. It follows from the closed graph theorem that the bases $\{x_n\}$ and $\{y_n\}$ are equivalent if and only if the map $T: X \rightarrow Y$ defined by $T(\sum_{i=1}^{\infty} t_i x_i) = \sum_{i=1}^{\infty} t_i y_i$ is an isomorphism from X onto Y and therefore two mutually equivalent bases are both seminormalized if one of them is. Assume that $\{x_n\}$ is a basis of X and that $\{n_k\}$ is an increasing sequence of positive integers then the sequence $\{x_{n_k}\}$ is called a *subbasis* of $\{x_n\}$. It is easy to see that the subbasis $\{x_{n_k}\}$ forms a basis for the subspace it spans in X . A subbasis $\{x_{n_k}\}$ is said to be *complemented* if $\sum_{k=1}^{\infty} t_{n_k} x_{n_k}$ converges whenever $\sum_{n=1}^{\infty} t_n x_n$ is convergent. We call a basis $\{x_n\}$ *unconditional* if for any choice of $\{\epsilon_i\}$ such that $|\epsilon_i| = 1$, $\sum_{i=1}^{\infty} \epsilon_i t_i x_i$ converges whenever $\sum_{i=1}^{\infty} t_i x_i$ converges. It is clear that every subbasis of an unconditional basis is complemented.

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Let $\mathcal{Q}$ be a family of bases (not necessarily in the same space). Following [3] we call a basis $\{x_n\}$ [complementably] universal with respect to $\mathcal{Q}$ if any member of $\mathcal{Q}$ is equivalent to a [complemented] subspace of $\{x_n\}$. A. Pełczyński proved the following

PROPOSITION 1 (THEOREM 1 OF [3]). *The following families of bases contain complementably universal elements:*

- (a) *the family of all seminormalized bases;*
- (b) *the family of all seminormalized unconditional bases.*

It is the first part of Proposition 1 which we are going to use as our main tool.

2. Let S denote the set of all sequences $\{t_n\}$ of scalars with only finitely many t_i different from 0 and let $\phi = \phi(t_1, t_2, \dots)$ be a real function defined on S . We call ϕ a *norming function* if the following conditions are satisfied for every scalar α and any two sequences $\{t_n\}$ and $\{s_n\}$ in S :

- (1) $\phi(t_1, t_2, \dots) \geq 0$ and $\phi(0, \dots, 0, 1, 0, \dots) = 1$ for all n .
(n zeros)
- (2) $\phi(t_1, t_2, \dots) = 0$ if and only if $t_i = 0$ for all i .
- (3) $\phi(\alpha t_1, \alpha t_2, \dots) = |\alpha| \cdot \phi(t_1, t_2, \dots)$.
- (4) $\phi(s_1 + t_1, s_2 + t_2, \dots) \leq \phi(s_1, s_2, \dots) + \phi(t_1, t_2, \dots)$.
- (5) $\phi(t_1, t_2, \dots, t_n, 0, 0, \dots) \leq \phi(t_1, t_2, \dots, t_n, t_{n+1}, 0, 0, \dots)$
for all n .

LEMMA 1. *Let $\{x_n\}$ be a normalized basis of a Banach space X , let $\phi = \phi(t_1, t_2, \dots)$ be a norming function and put*

$$\Phi(\{s_n\}) = \sup_n \phi(s_1, s_2, \dots, s_n, 0, 0, \dots)$$

for any sequence $\{s_n\}$ of scalars. Let $X_\phi = \{ \sum_{i=1}^\infty t_i x_i \in X \mid \Phi(\{t_n\}) < \infty \}$.

Then X_ϕ is a Banach space under the norm defined for $x = \sum_{i=1}^\infty t_i x_i$ by $\|x\| = \max\{\|x\|, \Phi(\{t_n\})\}$.

PROOF. It is obvious that X_ϕ is a normed space. To prove completeness, let $y_k = \sum_{i=1}^\infty \alpha_i^k x_i$ be a Cauchy sequence in X_ϕ . Then, clearly, $\{y_k\}$ is a Cauchy sequence in X and hence has a limit $y = \sum_{i=1}^\infty \alpha_i x_i$ in X . As is well known (see e.g. [2, p. 69 (4)]) there is an $M \geq 1$ such that $\|\sum_{i=1}^n \alpha_i x_i\| \leq M \|\sum_{i=1}^{n+k} \alpha_i x_i\|$ for all natural n and k and every sequence $\alpha_1, \alpha_2, \dots, \alpha_{n+k}$ of scalars. Given $\epsilon > 0$ choose $k = k(\epsilon)$ such

that $\|y_p - y_q\| < M^{-1} \cdot \epsilon$ whenever $p, q > k$. It follows that for each n

$$\left\| \sum_{i=1}^n (\alpha_i^p - \alpha_i^q) x_i \right\| \leq M \cdot \max \left\{ \left\| \sum_{i=1}^{\infty} (\alpha_i^p - \alpha_i^q) x_i \right\|, \Phi(\{\alpha_i^p - \alpha_i^q\}) \right\} \leq \epsilon.$$

Letting q tend to ∞ we get

$$(6) \quad \left\| \sum_{i=1}^n (\alpha_i^p - \alpha_i) x_i \right\| \leq \epsilon \quad \text{if } p > k.$$

Since $\left\| \sum_{i=1}^{\infty} (\alpha_i^p - \alpha_i) x_i \right\| \leq \sup_n \left\| \sum_{i=1}^n (\alpha_i^p - \alpha_i) x_i \right\|$ it follows from (6) that $\left\| \sum_{i=1}^{\infty} (\alpha_i^p - \alpha_i) x_i \right\| \leq \epsilon$, which implies that $y = \sum_{i=1}^{\infty} \alpha_i x_i \in X_\phi$ and that $\|y_k - y\| \xrightarrow{k \rightarrow \infty} 0$. This completes the proof.

Given a normalized basis $\{x_n\}$ in a Banach space X , of particular interest is the norming function ϕ defined for each $\{t_n\} \in S$ by

$$(7) \quad \phi(t_1, t_2, \dots, t_n, \dots) = \sup_{\mu \in \Sigma} \left\| \sum_{i \in \mu} t_i x_i \right\|$$

where Σ is the collection of all finite subsets of positive integers. By Lemma 1, X_ϕ is a Banach space. Let us denote by $\bar{x}_n$ the element x_n when regarded as a member of X_ϕ , then $\{\bar{x}_n\}$ is an unconditional basis of X_ϕ . Indeed, as is well known (see e.g. [2, p. 59]) a series $\sum_{i=1}^{\infty} t_i \bar{x}_i$ converges unconditionally if and only if

$$\sup_{\mu \in \Sigma} \left\| \sum_{i \in \mu} t_i \bar{x}_i \right\| < \infty$$

but since

$$\sup_{\mu \in \Sigma} \left\| \sum_{i \in \mu} t_i \bar{x}_i \right\| = \sup_{\mu \in \Sigma} \left\| \sum_{i \in \mu} t_i x_i \right\| = \Phi(\{t_n\})$$

and this is a finite number whenever $\sum_{i=1}^{\infty} t_i x_i \in X_\phi$ we finally get that $\{\bar{x}_n\}$ is an unconditional basis. Note that X_ϕ consists of all elements $x = \sum_{i=1}^{\infty} t_i x_i$ of X for which the series converges unconditionally and therefore some properties of the space X_ϕ may show the "extent of unconditionality" of the basis $\{x_n\}$ of X .

The above observations lead us to the following

ALTERNATIVE PROOF OF PROPOSITION 1(b). Let $\{x_n\}$ be a normalized universal basis for the family of all seminormalized bases. (The existence of such a basis may be assumed by Proposition 1(a).) Let ϕ be defined by (7) and consider the basis $\{\bar{x}_n\}$ of X_ϕ as above. We have shown that $\{\bar{x}_n\}$ is unconditional and we now claim that it forms a universal basis for all seminormalized unconditional bases.

Indeed, let $\{y_k\}$ be a seminormalized unconditional basis. Then, by universality of $\{x_n\}$, there is a subbasis $\{x_{n_k}\}$ equivalent to $\{y_k\}$. Since $\|x\| \geq \|x\|$ for all $x \in X_\phi$, the convergence of $\sum_{k=1}^{\infty} t_k \bar{x}_{n_k}$ in X_ϕ implies the convergence of $\sum_{k=1}^{\infty} t_k x_{n_k}$ in X and hence, by equivalence, also $\sum_{k=1}^{\infty} t_k y_k$ converges. Conversely, if $\sum_{k=1}^{\infty} t_k y_k$ converges it converges unconditionally and, by equivalence, $\sum_{k=1}^{\infty} t_k x_{n_k}$ converges unconditionally in X . It follows that $\sum_{k=1}^{\infty} t_k x_{n_k} \in X_\phi$ which implies the convergence of $\sum_{k=1}^{\infty} t_k \bar{x}_{n_k}$ in X_ϕ . We have shown that there is a subbasis $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ which is equivalent to $\{y_k\}$ and thus the universality of $\{\bar{x}_n\}$ is proved.

A norming function ϕ is called *subsequence homogeneous* if, for any $\{t_n\} \in S$, $\phi(t_1, t_2, \dots) = \phi(s_1, s_2, \dots)$ for every sequence $\{s_n\} \in S$ having the same number, k say, of nonzero terms such that for each $1 \leq i \leq k$ the i th nonzero term of $\{s_n\}$ is equal to the i th nonzero term of $\{t_n\}$. The following result independently observed by Pełczyński shows the existence of universal members in several families of bases:

PROPOSITION 2. *Let ϕ be a subsequence homogeneous norming function and let $\mathcal{Q}_\phi$ denote the family of all seminormalized bases $\{y_k\}$ such that the convergence of $\sum_{k=1}^{\infty} t_k y_k$ implies that*

$$\Phi(\{t_n\}) = \sup_n \phi(t_1, t_2, \dots, t_n, 0, 0, \dots) < \infty.$$

Then $\mathcal{Q}_\phi$ contains a normalized universal member.

PROOF. Let $\{x_n\}$ be a normalized basis (in some space X) which is universal with respect to the family of all seminormalized bases. Construct the space X_ϕ as in Lemma 1 and denote by $\bar{x}_n$ the element x_n when regarded as a member of X_ϕ , then, as is easy to see, $\|\bar{x}_n\| = 1$ and $\{\bar{x}_n\} \in \mathcal{Q}_\phi$. Let $\{z_k\} \in \mathcal{Q}_\phi$ and let $\{x_{n_k}\}$ be a subbasis of $\{x_n\}$ (in X) equivalent to $\{z_k\}$. If $\sum_{k=1}^{\infty} t_k z_k$ converges then $\Phi(\{t_k\}) < \infty$ and, because ϕ is subsequence homogeneous, $\sum_{k=1}^{\infty} t_k x_{n_k} \in X_\phi$ and therefore $\sum_{k=1}^{\infty} t_k \bar{x}_{n_k}$ converges in X_ϕ . Conversely, if $\sum_{k=1}^{\infty} t_k \bar{x}_{n_k}$ converges in X_ϕ then $\sum_{k=1}^{\infty} t_k x_{n_k}$ converges in X (note that $\|x\| \leq \|x\|$ for all $x \in X_\phi$) and, by equivalence, $\sum_{k=1}^{\infty} t_k z_k$ is convergent. It follows that $\{z_k\}$ is equivalent to $\{\bar{x}_{n_k}\}$ in X_ϕ , and this proves the universality of the basis $\{\bar{x}_n\}$ of X_ϕ .

REMARK 1. In a private communication A. Pełczyński remarked that by using Proposition 1(a), Proposition 2 and duality one can prove the existence of a universal member in the family $\mathcal{Q}_\phi^*$ of all bases $\{y_n\}$ such that $\sum \alpha_i y_i$ converges whenever $\Phi(\{\alpha_n\}) < \infty$.

3. A basis $\{x_n\}$ of a Banach space X is called *shrinking* if the biorthogonal functionals $\{x_n^*\}$ (defined by $x_n^*(x_k) = \delta_{n,k}$) span the

dual space X^* . It is called *boundedly complete* if $\sum_{k=1}^{\infty} t_k x_k$ converges whenever $\sup_n \|\sum_{k=1}^n t_k x_k\| < \infty$. A. Pełczyński proves [3, Theorem 4] that there is no shrinking basis universal for the family of all shrinking bases. It is conjectured in [3, p. 267] that a similar result holds for boundedly complete bases. Before verifying this conjecture we need the following

LEMMA 2. Let $\{x_n\}$ be a basis of a Banach space X and define ϕ by (7). Construct the space X_ϕ as before and denote by $\bar{x}_k$ the element x_k when regarded as an element of X_ϕ . Then $\{\bar{x}_n\}$ is a boundedly complete basis of X_ϕ if any one of the following conditions is satisfied:

- (a) $\{x_n\}$ is a boundedly complete basis of X .
- (b) X is isomorphic to a dual space Y^* .
- (c) X is weakly sequentially complete.
- (d) X does not have a subspace isomorphic to c_0 (=the space of all sequences $\lambda = \{\lambda_n\}$ of scalars converging to 0 with $\|\lambda\| = \max_n |\lambda_n|$).

PROOF. We first note that any one of the first three conditions implies (d). Indeed (a) $\Rightarrow$ (b) by [2, p. 67, Theorem 1, and p. 70, Lemma 2]. (b) $\Rightarrow$ (d) because by [1, Theorem 4] c_0 cannot be isomorphic to a subspace of a separable dual space. (c) $\Rightarrow$ (d) because c_0 is not weakly sequentially complete. Now assume that (d) holds. If $\{\bar{x}_n\}$ is not a boundedly complete basis of X_ϕ then by the proof of Theorem 2 of p. 74 of [2] there is a sequence of blocks

$$z_k = \sum_{i=p_k+1}^{p_{k+1}} \beta_i \bar{x}_i, \quad k = 1, 2, \dots,$$

such that $\{p_k\}$ is an increasing sequence of positive integers $\inf_k \|z_k\| = \delta > 0$ and $\{z_k\}$ is a basis (for its span) equivalent to the unit vectors of c_0 . It follows from the definition of the norm $\|\cdot\|$ that for each k there is a subset $\mu_k \subset \{p_k+1, p_k+2, \dots, p_{k+1}\}$ such that $\|\sum_{i \in \mu_k} \beta_i x_i\| \geq \delta/2$. We claim that the elements $u_k = \sum_{i \in \mu_k} \beta_i x_i$, $k = 1, 2, \dots$, form a basis (for the subspace they span in X) which is equivalent to the unit vector basis of c_0 . Proving this claim which obviously contradicts (d) we will establish the assertion of Lemma 2.

It is well known and easy to show that any sequence of blocks (like u_k) with respect to a given basis of a Banach space X forms a basis for the subspace it spans in X . It is also clear that in view of the inequality $\inf_k \|u_k\| \geq \delta/2$ the convergence of $\sum_{i=1}^{\infty} \alpha_i u_i$ in X implies that $\alpha_i \xrightarrow{i \rightarrow \infty} 0$. To prove the converse, let $\alpha_i \xrightarrow{i \rightarrow \infty} 0$. Then $\sum_{i=1}^{\infty} \alpha_i z_i$ converges in X_ϕ (because of the equivalence of $\{z_k\}$ to the unit vector basis of c_0). It follows from the definition of the norm

$\|\cdot\|$ that for any $x \in X_\phi$ $\|x\| \geq \|x\|$ and also that $\|\sum_{i \in \nu} \gamma_i \tilde{x}_i\| \geq \|\sum_{i \in \mu} \gamma_i \tilde{x}_i\|$ for any $\mu \subset \nu \in \Sigma$ and every $\sum_{i \in \nu} \gamma_i \tilde{x}_i \in X_\phi$. Therefore for any $m < n$,

$$\begin{aligned} \left\| \sum_{i=m}^n \alpha_i z_i \right\| &= \left\| \sum_{i=m}^n \alpha_i \left(\sum_{j=p_i+1}^{p_{i+1}} \beta_j \tilde{x}_j \right) \right\| \\ &\geq \left\| \sum_{i=m}^n \alpha_i \left(\sum_{j \in \mu_i} \beta_j \tilde{x}_j \right) \right\| \geq \left\| \sum_{i=m}^n \alpha_i u_i \right\| \end{aligned}$$

and hence $\sum_{i=1}^\infty \alpha_i u_i$ converges in X . This concludes the proof of Lemma 2.

Let X and Y be Banach spaces and $1 \leq p \leq \infty$. Then $X \oplus_p Y$ denotes the space of all pairs (x, y) where $x \in X, y \in Y$, with norm

$$\begin{aligned} \|(x, y)\| &= (\|x\|^p + \|y\|^p)^{1/p} \quad \text{if } 1 \leq p < \infty, \\ &= \max\{\|x\|, \|y\|\} \quad \text{if } p = \infty. \end{aligned}$$

Following [3], given a countable family $\{X_\alpha\}_{\alpha \in A}$ of Banach spaces we denote by $(P_{\alpha \in A} X_\alpha)_2$ the space of all functions $x = x(\alpha)$ on A with $x(\alpha) \in X_\alpha$ for every $\alpha \in A$ such that $\|x\| = (\sum_{\alpha \in A} \|x(\alpha)\|^2)^{1/2} < \infty$. Let ω_1 denote the first uncountable ordinal and, as in [3], construct by transfinite induction a family of spaces X_α for all ordinals $\alpha < \omega_1$ as follows: $X_0 = l_2$, $X_\alpha = X_\beta \oplus_2 l_1$ if $\alpha = \beta + 1$ and $X_\alpha = (P_{\beta < \alpha} X_\beta)_2$ if α is a limit. It is easy to prove that, for each $\alpha < \omega_1$, X_α is a reflexive Banach space having a normalized unconditional basis $\{u_{\alpha,k}\}_{k=1}^\infty$ which is both shrinking and boundedly complete (see e.g. [2 p. 71]). Moreover, for each $\alpha < \omega_1$ the biorthogonal functionals $\{u_{\alpha,k}^*\}_{k=1}^\infty$ of $\{u_{\alpha,k}\}_{k=1}^\infty$ form a basis for X_α^* sharing the same properties. In [3] Pelczyński is using the following result of Szlenk [4].

Let F be a Banach space which contains for each $\alpha < \omega_1$ a subspace isomorphic to X_α . Then F is not separable.

This result enables us to prove Pelczyński's conjecture namely

THEOREM. *The following families of bases do not contain a universal member:*

- The family of all seminormalized boundedly complete bases;*
- The family of all seminormalized bases of spaces isomorphic to dual spaces;*
- The family of all seminormalized bases of weakly sequentially complete spaces;*
- The family of all seminormalized bases of spaces which do not contain a subspace isomorphic to c_0 .*

PROOF. Suppose that a basis $\{z_n\}$ (not necessarily seminormalized) in some Banach space X is universal for one of the above mentioned families and belongs to this family. Put $x_n = z_n/\|z_n\|$, then the basis $\{x_n\}$ need not be universal with respect to the whole family, $\mathcal{A}$ say, under consideration however, it is universal for all unconditional members of $\mathcal{A}$. (This follows easily from the fact that if $\{y_n\}$ is an unconditional basis and $\{\alpha_n\}$ is a sequence of scalars with $0 < \inf |\alpha_n| \leq \sup |\alpha_n| < \infty$ then $\{\alpha_n y_n\}$ is a basis equivalent to $\{y_n\}$.) Define ϕ by (7) and construct the space X_ϕ and the unconditional basis $\{\bar{x}_n\}$ as in §2. We know that a series $\sum_{i=1}^{\infty} t_i \bar{x}_i$ converges in X_ϕ if and only if it converges unconditionally in X and hence for each unconditional member $\{y_n\}$ of $\mathcal{A}$ there is a subbasis of $\{\bar{x}_n\}$ equivalent to $\{y_n\}$. In particular, for each $\alpha < \omega_1$ the basis $\{u_{\alpha,k}^*\}_{k=1}^{\infty}$ of X_α^* being a member of $\mathcal{A}$ is equivalent to some subbasis $\{\bar{x}_{n(\alpha,k)}^*\}_{k=1}^{\infty}$ of $\{\bar{x}_n\}$. It follows from (7) and Lemma 2 that $\{\bar{x}_n\}$ is a boundedly complete monotone basis and therefore, by [2, p. 70, Lemma 2], the biorthogonal functionals $\{\bar{x}_n^*\}$ of $\{\bar{x}_n\}$ form an unconditional shrinking basis of the subspace F they span in X_ϕ^* and F^* is isometric to X_ϕ . Since every subbasis of an unconditional basis is complemented it is easy to see that for each $\alpha < \omega_1$ the subbasis $\{\bar{x}_{n(\alpha,k)}^*\}_{k=1}^{\infty}$ of $\{\bar{x}_n^*\}$ is equivalent to the basis $\{u_{\alpha,k}^*\}_{k=1}^{\infty}$ of X_α and hence F contains a subspace isomorphic to X_α . In view of Szlenk's result this contradicts the separability of $X_\phi = F^*$ and the Theorem is thus proved.

REMARK 2. Using entirely different methods P. Wojtaszczyk [5] has recently proved the Theorem for the family (a).

REFERENCES

1. C. Bessaga and A. Pelczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164. MR **22** #5872.
2. M. M. Day, *Normed linear spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 21, Springer-Verlag, Berlin, 1958. MR **20** #1187.
3. A. Pelczyński, *Universal bases*, Studia Math. **32** (1969), 247–268.
4. W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. **30** (1968), 53–61. MR **37** #3327.
5. P. Wojtaszczyk, *On separable Banach spaces containing all separable reflexive Banach spaces*, Studia Math. (to appear).

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