PERIODIC SOLUTIONS OF LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT[1]

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ABSTRACT. This paper is concerned with the question of the existence of periodic solutions of periodic linear second order differential equations with deviating argument. Using a fixed point theorem for multivalued mappings and results concerning boundary value problems for such equations, we prove that the existence of periodic solutions of both types of differential inequalities implies the existence of periodic solutions. This result, in turn, is used to obtain the existence of periodic solutions of certain nonlinear differential equations with deviating argument.

1. Consider the second order differential equation with deviating argument

$$Lx(t) \equiv x''(t) + a(t)x'(t) + b(t)x(t) + c(t)x(t - d(t)) = e(t), \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t)$ are continuous real-valued functions which are periodic of period $T>0$, and $c(t) \geqq 0$. (No restrictions on the sign of $d(t)$ are made.) In this paper we give sufficient conditions under which equation (1) has a solution $x(t) \in C^2(-\infty, \infty)$ which is periodic of period T. Our main theorem, which is a mean value type theorem for the operator L, takes the following form.

THEOREM 1. *Let there exist functions $\alpha(t), \beta(t) \in C^2(-\infty, \infty)$ which are periodic of period T, such that*

$$\alpha(t) \leqq \beta(t) \quad \text{and} \quad L\beta(t) \leqq e(t) \leqq L\alpha(t).$$

Then there exists a periodic solution $x(t)$ of (1) such that $\alpha(t) \leqq x(t) \leqq \beta(t)$.

The theorem is proved by using existence results for boundary value problems for nonlinear second order differential equations with deviating arguments established in [2] and [3] and a special case of a fixed point theorem for multivalued maps due to Eilenberg and Montgomery [1]. The proof proceeds via several lemmas given in the next section.

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2. Let P denote the set of all real continuous functions of period T and for $\phi \in P$, let

$$\|\phi\| = \max_{0 \leqq t \leqq T} |\phi(t)|.$$

Then $(P, \|\cdot\|)$ is a Banach space (P, of course may be identified with that subspace of $C[0, T]$ consisting of all those continuous functions $\psi(t)$ such that $\psi(0) = \psi(T)$). If $\alpha, \beta \in P$ such that $\alpha(t) \leqq \beta(t)$, $0 \leqq t \leqq T$, we denote by $[\alpha, \beta]$ the set

$$[\alpha, \beta] = \{\phi \in P : \alpha(t) \leqq \phi(t) \leqq \beta(t), t \in [0, T]\}.$$

Consider the following boundary value problem

$$Lx(t) = e(t), \quad t \in [0, T], \tag{2}$$

$$x(t) = \phi(t), \quad t \notin (0, T), \tag{3}$$

where $\phi \in P$.

As a special case of Theorem 9 of [**3**] we have the following:

LEMMA 1. *Let $\alpha, \beta \in P$ such that $\alpha(t) \leqq \beta(t)$ and $L\beta(t) \leqq e(t) \leqq L\alpha(t)$, $t \in [0, T]$. Then for any $\phi \in [\alpha, \beta]$ there exists a solution $x(t)$ of* (2), (3) *such that $x \in [\alpha, \beta]$. Furthermore there exists a constant N depending on α, β and L such that $|x'(t)| \leqq N$, $t \in [0, T]$.*

REMARK. We point out that a solution of (2), (3) is a solution of (1) only on the interval $[0, T]$, outside that interval it coincides with the preassigned function ϕ.

Let α and β be as in Lemma 1. For each $\phi \in [\alpha, \beta]$ denote by $S(\phi)$ the set of all solutions $x(t)$ of (2), (3) such that $x \in [\alpha, \beta]$.

LEMMA 2. *For each $\phi \in [\alpha, \beta]$, $S(\phi)$ is a convex subset of $[\alpha, \beta]$.*

PROOF. This follows from the linearity of equation (1).

LEMMA 3. *The multivalued map S is continuous in the following sense: If $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in $[\alpha, \beta]$ such that $\|\phi_n - \phi\| \to 0$ and $\|\psi_n - \psi\| \to 0$ as $n \to \infty$, $\psi_n \in S(\phi_n)$, then $\psi \in S(\phi)$.*

PROOF. By Lemma 1, the set $\bigcup\{S(\phi) : \phi \in [\alpha, \beta]\}$ is a uniformly bounded and equicontinuous subset of $[\alpha, \beta]$. Since ψ_n is a solution of the boundary value problem

$$Lx(t) = e(t), \quad t \in [0, T], \qquad x(t) = \phi_n(t), \quad t \notin (0, T),$$

it follows that $\{\psi_n'\}$ is also uniformly bounded and equicontinuous on $[0, T]$. It now easily follows that $\psi \in S(\phi)$.

Define the set $M(r)$ by

$$M(r) = \{\phi \in [\alpha, \beta]: \phi \in C^1[0, T], \phi(0) = \mid \phi'(t) \mid \leqq N, t \in [0, T]\}.$$

$M(r)$ is a compact convex subset of P and by Lemma 1, $S(\phi) \subset M(r)$ for every $\phi \in M(r)$.

LEMMA 4. *The multivalued map $S: M(r) \to M(r)$ has a fixed point in $M(r)$, i.e. there exists $\phi \in M(r)$ such that $\phi \in S(\phi)$.*

PROOF. This follows from Lemmas 1, 2, 3, the fixed point theorem of Eilenberg and Montgomery [**1**, Theorem 1] and the observation that every compact convex subset of a Banach space is an absolute neighborhood retract and is acyclic (see Lefschetz [**4**, p. 119, proof of 29.1]).

It is now an easy matter to verify that there exists r, $\alpha(0) \leqq r \leqq \beta(0)$, such that some fixed point ϕ of the mapping S on $M(\bar{r})$ has the property that its periodic extension is a solution of (1) on $(-\infty, \infty)$. This completes the proof of Theorem 1.

REMARK. If it is known that boundary value problems of type (2), (3) have at most one solution, we may, of course, apply the Schauder fixed point theorem to conclude the existence of a periodic solution. However, in many cases (even if $d(t) \equiv 0$) uniqueness fails and moreover uniqueness in general is difficult to check particularly when $d(t)$ changes sign, i.e. when (1) is an equation of neutral type.

As an application of Theorem 1, consider the following equation

$$x''(t) + a(t)x'(t) - bx(t) + cx(t - d(t)) = e(t), \tag{4}$$

where, as before, $a(t)$, $d(t)$ and $e(t)$ are continuous and periodic of period T and b and c are positive constants such that $b > c$. In this case, we may choose β to be a positive constant such that

$$(c - b)\beta \leqq e(t), \quad t \in [0, T], \tag{5}$$

and α negative such that

$$(c - b)\alpha \geqq e(t), \quad t \in [0, T]. \tag{6}$$

We conclude that there exists a periodic solution $x(t)$ of (4) such that $\alpha \leqq x(t) \leqq \beta$.

3. In applying Lemma 1 to equation (4), we note that the constant N is also independent of the deviating argument $d(t)$. This observation makes it possible to conclude the existence of a periodic solution of the nonlinear equation

$$Lx(t) = x''(t) + a(t)x'(t) - bx(t) + cx(t - d(t, x(t))) = e(t). \tag{7}$$

THEOREM 2. *Let $a(t)$ and $e(t)$ be continuous functions which are periodic of period T and let b and c be positive constants with $b>c$. Let β and α be as in* (5) *and* (6) *and let $d(t, x)$ be periodic in t of period T and continuous in (t, x), $\alpha \leqq x \leqq \beta$. Then there exists a periodic solution of* (7) *such that $\alpha \leqq x(t) \leqq \beta$.*

PROOF. Let $M=\{\phi\in[\alpha, \beta]:\phi\in C^2(-\infty, \infty), |\phi'(t)| \leqq N\}$. Then M is a compact convex subset of P. For each $\phi\in M$ consider the equation

$$x''(t) + a(t)x'(t) - bx(t) + cx(t - d(t, \phi(t))) = e(t). \tag{8}$$

By Theorem 1, there exists a periodic solution $x(t)$ of (8) such that $x\in M$. Denote by $S(\phi)$ the set of all such periodic solutions of (8). As before one may easily verify that the multivalued map S has a fixed point. Fixed points of S are solutions of (7).

REMARK. Several variations of Theorem 2 are possible, one may for example replace $e(t)$ by a possibly nonlinear function $e(t, x)$, periodic in t of period T, continuous in (t, x) and bounded. Also the operator L in (1) and (7) may contain several different terms containing a deviating argument as long as the coefficient of each such term is nonnegative.

Theorems 1 and 2 may also be extended to the nonlinear equations

$$x''(t)=f(t, x(t), x'(t), x(t-d(t)))$$

and

$$x''(t)=f(t, x(t), x'(t), x(t-d(t, x(t)))),$$

provided we make the additional assumption that boundary value problems for such equations have at most one solution.

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