AN EXAMPLE CONCERNING CORE MEASURES

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ABSTRACT. We shall construct a uniform algebra having a multiplicative linear functional whose space of representing measures is infinite-dimensional but nevertheless has a core measure. This answers in the negative an open question in Gamelin's book [1, p. 106].

Let $\Delta = \{Z \in \mathbb{C} : |Z| \leq 1\}$, the closed unit disc, and let T be its boundary. The disc algebra A consists of those $f \in C(T)$ which have extensions $\hat{f} \in C(\Delta)$ analytic on the interior of Δ . We assume a basic familiarity with this algebra. In particular, if $Z \in \Delta$ and μ is a positive Borel measure on T such that $\int f d\mu = \hat{f}(Z)$ for all $f \in A$, then either μ is point mass at Z (if $Z \in T$) or $\mu = P\lambda$ where λ is normalized Lebesgue measure on T and P is the Poisson kernel for Z (if $Z \in \Delta/T$). This information, and the notions of representing measure and core measure, are in [1].

For $0 \leq n \leq \infty$ let $T_n = T \times \{p_n\} \subset \mathbb{C}^2$ where $p_0 = 2$, $p_\infty = 0$ and $p_n = 1/n$, $1 \leq n < \infty$. Let $X = \bigcup \{T_n : 0 \leq n \leq \infty\}$, a compact set. Let $\eta_n : T \rightarrow X$ be $\eta_n(Z) = (Z, p_n)$. If $f \in C(X)$ let $f_n = f \circ \eta_n \in C(T)$. If μ is a finite Borel measure on X , let $\mu_n = \mu \circ \eta_n$, a finite Borel measure on T . Then $\sum_{0 \leq n \leq \infty} \|\mu_n\| = \|\mu\|$ and, for each $f \in C(X)$,

$$(1) \quad \int f d\mu = \sum_{0 \leq n \leq \infty} \int f_n d\mu_n.$$

This effects a one-to-one linear correspondence between measures on X and sequences $\{\mu_n\}$ of measures on T satisfying $\sum_{0 \leq n \leq \infty} \|\mu_n\| < \infty$. Note also that $0 \leq \mu \leq C\nu$ if and only if $0 \leq \mu_n \leq C\nu_n$ for all n , where C is a nonnegative constant.

Fix a sequence (to be specified later) $\{Z_n\}_{1 \leq n < \infty}$ of distinct complex numbers with $0 < |Z_n| < 1$ and $Z_n \rightarrow 1$, and let

$$\mathfrak{A} = \{f \in C(X) : f_n \in A, 0 \leq n \leq \infty, \text{ and } \hat{f}_n(Z_n) = \hat{f}_0(Z_n), 1 \leq n < \infty\}.$$

Clearly $\mathfrak{A}$ is a uniformly closed subalgebra of $C(X)$ containing the constants. The functions we will construct below show that $\mathfrak{A}$

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separates points on X except for the pair $\{(1, 0), (1, 2)\}$, so is a uniform algebra on the quotient space Y obtained from X by identifying this pair to a point. Let ϕ be the multiplicative linear functional $\phi(f) = f_0(0)$ on $\mathfrak{A}$, and let M_ϕ denote the (convex, weak* compact) set of positive Borel measures μ on X representing ϕ , i.e., $\int f d\mu = \phi(f)$ for all $f \in \mathfrak{A}$. Fix $\mu \in M_\phi$.

Suppose $F \in A$ and $\hat{F}(Z_N) = 0$ for some N , $1 \leq N < \infty$. Define $f \in \mathfrak{A}$ by $f_N = F$ and $f_n = 0$, $n \neq N$. Then by (1) $\int F d\mu_N = \int f d\mu = \phi(f) = 0$. Thus there is a constant $C_N \geq 0$ such that $\mu_N = C_N P_N \lambda$ where P_N is the Poisson kernel for Z_N .

Next suppose $F \in A$ and $F(1) = 0$. Define $f \in \mathfrak{A}$ by $f_0 = 0$, $f_\infty = F$ and $f_n = F - \hat{F}(Z_n)$, $1 \leq n < \infty$. By (1) $\int F d\mu_\infty = \int f d\mu = \phi(f) = 0$, so there is a constant $C_\infty \geq 0$ such that $\mu_\infty = C_\infty \delta$ where δ is point mass at 1.

Finally suppose $F \in A$ and define $f \in \mathfrak{A}$ by $f_n = F$ for all n . By (1) $\hat{F}(0) = \phi(f) = \int f d\mu = \sum_{0 \leq n \leq \infty} \int F d\mu_n = \int F d(\sum_{0 \leq n \leq \infty} \mu_n)$, so $\sum_{0 \leq n \leq \infty} \mu_n = \lambda$. Thus $C_\infty = 0$ and we see that the form of the sequence $\{\mu_n\}$ is

$$\begin{aligned} \mu_n &= C_n P_n \lambda, & 1 \leq n < \infty, \\ (2) \quad \mu_0 &= \left(1 - \sum_{1 \leq n < \infty} C_n P_n\right) \lambda, \\ \mu_\infty &= 0, \end{aligned}$$

where $\{C_n\}_{1 \leq n < \infty}$ is a sequence of nonnegative numbers satisfying

$$(3) \quad \sum_{1 \leq n < \infty} C_n P_n \leq 1.$$

Conversely, it is easy to see that any measure given by (2), (3) is in M_ϕ . Note that M_ϕ consists of measures which assign no mass to points, hence may be identified with the space of representing measures for $\mathfrak{A}$ on Y .

Let $\gamma_n = \max P_n \geq 1$, $1 \leq n < \infty$. If for $1 \leq N < \infty$ we define μ^N by (2) with $C_N = \gamma_N^{-1}$ and $C_n = 0$, $n \neq N$, then the μ^N are affinely independent members of M_ϕ , so the latter is infinite dimensional.

It remains to select the Z_n so that M_ϕ has a core measure. Fix a sequence $\{\theta_n\}_{1 \leq n < \infty}$ of distinct nonzero numbers in $(-\pi, \pi)$ with $\theta_n \rightarrow 0$. Let $\{I_n\}_{1 \leq n < \infty}$ be a sequence of disjoint open intervals in $(-\pi, \pi)$ with $\theta_n \in I_n$. Let $\{\alpha_n\}_{1 \leq n < \infty}$ be a sequence of positive numbers such that $\sum_{1 \leq n < \infty} \alpha_n < \infty$. The form of the Poisson kernel shows that if we take $r_n > 0$ less than but close enough to 1 and set $Z_n = r_n e^{i\theta_n}$ then

$$(4) \quad P_n < \alpha_n \quad \text{on } [-\pi, \pi] \setminus I_n.$$

Finally, select $\epsilon > 0$ so that

$$(5) \quad \sigma \equiv \epsilon \left(1 + \sum_{1 \leq n < \infty} \alpha_n \right) < 1.$$

Let $\tilde{C}_n = \epsilon \gamma_n^{-1}$, $1 \leq n < \infty$. Since any $\theta \in [-\pi, \pi]$ is in at most one I_n and since $\gamma_n \geq 1$, (4) leads to

$$(6) \quad \sum_{1 \leq n < \infty} \tilde{C}_n P_n \leq \sigma.$$

So in view of (5) and (3), we can define $\tilde{\mu} \in M_\phi$ by (2) with $C_n = \tilde{C}_n$.

Let $\mu \in M_\phi$ be given by the sequence $\{C_n\}_{1 \leq n < \infty}$. By (3), $C_n \leq \gamma_n^{-1} = \epsilon^{-1} \tilde{C}_n$, so $\mu_n \leq \epsilon^{-1} \tilde{\mu}_n$, $1 \leq n < \infty$. By (2) $\mu_0 \leq \lambda$ while by (2), (5), (6) $(1 - \sigma)\lambda \leq \tilde{\mu}_0$; thus $\mu_0 \leq (1 - \sigma)^{-1} \tilde{\mu}_0$. If we set $K = \max\{\epsilon^{-1}, (1 - \sigma)^{-1}\}$, we see that $\mu \leq K\tilde{\mu}$. By [1, IV. 5.1] $\tilde{\mu}$ is a core measure for M_ϕ .

REMARKS. Examination of our estimates shows that if we preassign a constant $K > 2$, we can arrange that $\mu \leq K\tilde{\mu}$ for all $\mu \in M_\phi$. So if $0 < \tau < 1/2$ is preassigned, we may arrange that $\tilde{\mu} - \tau\nu \in M_\phi$ whenever $\tilde{\mu} + \nu \in M_\phi$.

It is also worth noting that every multiplicative positive measure for $\mathfrak{A}$ on Y is an Arens-Singer measure (in fact, $\exp(\mathfrak{A}) = \mathfrak{A}^{-1}$); on the other hand, M_ϕ contains but one Jensen measure (namely the case $C_n = 0$, all n), and it is not a core measure.

REFERENCES

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