CONGRUENCE RELATIONS IN DIRECT PRODUCTS

GRANT A. FRASER AND ALFRED HORN¹

ABSTRACT. This paper studies conditions under which every congruence relation θ in a direct product $A \times B$ of algebras is of the form $\theta_1 \times \theta_2$, where θ_1 and θ_2 are congruence relations in A and B respectively. It is shown that for any equational class K , every such θ in every $A \times B$ in K has this property if and only if K satisfies certain identities.

It is well known that if A and B are rings with unit element, then every ideal in $A \times B$ is of the form $I_1 \times I_2$, where I_1, I_2 are ideals in A and B respectively. In [1, Problem 40], G. Grätzer asks to characterize those equational classes K of algebras such that for any $A, B \in K$, every congruence relation θ in $A \times B$ is of the form $\theta_1 \times \theta_2$, where θ_1, θ_2 are congruence relations in A and B respectively.² In Theorem 5 below, we shall show that K can be characterized by Malcev-type conditions, that is, by the existence of certain identities in K .

We restrict our attention to algebras with finitary operations (possibly infinite in number). The set of congruence relations in A is denoted by $C(A)$. If $\theta \in C(A)$, then $a_1 \theta a_2$ denotes $(a_1, a_2) \in \theta$. If $\theta_1 \in C(A)$ and $\theta_2 \in C(B)$, let $\theta_1 \times \theta_2$ be the relation $\{((a_1, b_1), (a_2, b_2)) : a_1 \theta_1 a_2 \text{ and } b_1 \theta_2 b_2\}$. Clearly $\theta_1 \times \theta_2 \in C(A \times B)$. Let $I_A = \{(a, a) : a \in A\}$ be the identity relation on A , and $U_A = A \times A$ be the universal relation on A . If $a_1, a_2 \in A$, let $\theta(a_1, a_2)$ be the smallest congruence relation on A which contains (a_1, a_2) .

In $C(A \times B)$, we let Π_1 denote the kernel of the projection on A . Clearly $\Pi_1 = \{((a_1, b_1), (a_2, b_2)) : a_1 = a_2\} = I_A \times U_B$; similarly let $\Pi_2 = U_A \times I_B$. It is well known (see [1]) that $C(A)$ is a lattice in which $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$, and

$$\theta_1 \vee \theta_2 = \bigcup_{n < \omega} (\rho_0 \circ \rho_1 \circ \cdots \circ \rho_n),$$

where $\rho_i = \theta_1$ for even i , and $\rho_i = \theta_2$ for odd i .

LEMMA 1. *If $\rho_1, \theta_1 \in C(A)$ and $\rho_2, \theta_2 \in C(B)$, then*

$$(\rho_1 \times \rho_2) \vee (\theta_1 \times \theta_2) = (\rho_1 \vee \theta_1) \times (\rho_2 \vee \theta_2).$$

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² This problem has been solved independently by Tah-Kai Hu.

PROOF. It is obvious that $\rho_1 \times \rho_2$ and $\theta_1 \times \theta_2$ are contained in $(\rho_1 \vee \theta_1) \times (\rho_2 \vee \theta_2)$. Now let $(a, b) \in (\rho_1 \vee \theta_1) \times (\rho_2 \vee \theta_2)$. Then for some m , $a \in \phi_0 \circ \dots \circ \phi_m$ and $b \in \psi_0 \circ \dots \circ \psi_m$, where $\phi_i = \rho_1$, $\psi_i = \rho_2$ for even i , and $\phi_i = \theta_1$, $\psi_i = \theta_2$ for odd i (since $\phi_0 \circ \dots \circ \phi_n \subseteq \phi_0 \circ \dots \circ \phi_{n+1}$); hence

$$(a, b) \in (\phi_0 \circ \dots \circ \phi_m) \times (\psi_0 \circ \dots \circ \psi_m) = (\phi_0 \times \psi_0) \circ \dots \circ (\phi_m \times \psi_m) \\ \subseteq (\rho_1 \times \rho_2) \vee (\theta_1 \times \theta_2).$$

DEFINITION. If $\theta \in C(A \times B)$, we say θ has property P if there exist $\theta_1 \in C(A)$, $\theta_2 \in C(B)$ such that $\theta = \theta_1 \times \theta_2$. We say $A \times B$ has property P if every $\theta \in C(A \times B)$ has property P . If K is a class of similar algebras, we say K has property P if for all $A, B \in K$, $A \times B$ has property P .

It is easily seen that $A \times B$ has property P if and only if the map $(\theta_1, \theta_2) \rightarrow \theta_1 \times \theta_2$ is an isomorphism from $C(A) \times C(B)$ onto $C(A \times B)$.

THEOREM 1. Let $\theta \in C(A \times B)$. Then the following are equivalent:

- (1) θ has property P .
- (2) $\Pi_2 \cap (\Pi_1 \vee \theta) \subseteq \theta$ and $\Pi_1 \cap (\Pi_2 \vee \theta) \subseteq \theta$.
- (3) For all $a, a_1, a_2 \in A$ and all $b, b_1, b_2 \in B$, $(a_1, b_1)\theta(a_2, b_2)$ implies $(a_1, b)\theta(a_2, b)$ and $(a, b_1)\theta(a, b_2)$.

PROOF. (1) $\rightarrow$ (2): If $\theta = \theta_1 \times \theta_2$ then by Lemma 1,

$$\Pi_2 \cap (\Pi_1 \vee \theta) = \Pi_2 \cap ((I_A \times U_B) \vee (\theta_1 \times \theta_2)) \\ = (U_A \times I_B) \cap (\theta_1 \times U_B) = \theta_1 \times I_B \subseteq \theta,$$

and similarly $\Pi_1 \cap (\Pi_2 \vee \theta) \subseteq \theta$.

(2) $\rightarrow$ (3): Suppose $(a_1, b_1)\theta(a_2, b_2)$. Since $(a_1, b)\Pi_1(a_1, b_1)$ and $(a_2, b_2)\Pi_1(a_2, b)$ we have $(a_1, b)(\Pi_1 \circ \theta \circ \Pi_1)(a_2, b)$. Since $\Pi_1 \circ \theta \circ \Pi_1 \subseteq \theta \vee \Pi_1$, it follows that

$$(a_1, b)(\Pi_2 \cap (\theta \vee \Pi_1))(a_2, b)$$

and by (2), $(a_1, b)\theta(a_2, b)$. Similarly $(a, b_1)\theta(a, b_2)$.

(3) $\rightarrow$ (1): Let

$$\theta_1 = \{(a_1, a_2): \text{for some } b, (a_1, b)\theta(a_2, b)\}, \\ \theta_2 = \{(b_1, b_2): \text{for some } a, (a, b_1)\theta(a, b_2)\}.$$

Then by (3), $a_1\theta_1a_2$ implies $(a_1, b)\theta(a_2, b)$ for all $b \in B$. It is very easy to verify that $\theta_1 \in C(A)$, $\theta_2 \in C(B)$ and $\theta = \theta_1 \times \theta_2$.

COROLLARY 1. If $C(A \times B)$ is a distributive lattice, then $A \times B$ has property P .

PROOF. Since $\Pi_1 \cap \Pi_2 = I_{A \times B}$, it is easily seen that (2) holds.

Corollary 1 was pointed out to us by Alfred Hales. He gave a different proof of the sufficiency part of the next theorem, from which Corollary 1 also follows.

THEOREM 2. *If $\theta \in C(A \times B)$, then θ has property P if and only if*

$$(4) \quad (\Pi_1 \vee \theta) \cap (\Pi_2 \vee \theta) = \theta.$$

PROOF. If $\theta = \theta_1 \times \theta_2$, then by Lemma 1,

$$\begin{aligned} (\Pi_1 \vee \theta) \cap (\Pi_2 \vee \theta) &= [(I_A \times U_B) \vee (\theta_1 \times \theta_2)] \cap [(U_A \times I_B) \vee (\theta_1 \times \theta_2)] \\ &= [(I_A \vee \theta_1) \times (U_B \vee \theta_2)] \cap [(U_A \vee \theta_1) \times (I_B \vee \theta_2)] \\ &= (\theta_1 \times U_B) \cap (U_A \times \theta_2) = \theta_1 \times \theta_2 = \theta. \end{aligned}$$

Conversely, if (4) holds, then (2) is an immediate consequence.

THEOREM 3. *Let A and B be similar algebras. Then the following are equivalent:*

(5) *$A \times B$ has property P.*

(6) *For all $a_1, a_2 \in A$ and all $b_1, b_2 \in B$, $\theta(a_1, a_2) \times \theta(b_1, b_2) = \theta((a_1, b_1), (a_2, b_2))$.*

(7) *For all $a_1, a_2, a \in A$ and all $b_1, b_2, b \in B$, $((a_1, b), (a_2, b))$ and $((a, b_1), (a, b_2))$ are in $\theta((a_1, b_1), (a_2, b_2))$.*

PROOF. (5) $\rightarrow$ (6): Clearly, $((a_1, b_1), (a_2, b_2)) \in \theta(a_1, a_2) \times \theta(b_1, b_2)$. Suppose $((a_1, b_1), (a_2, b_2)) \in \theta \in C(A \times B)$. By (5), $\theta = \theta_1 \times \theta_2$ for some $\theta_1 \in C(A)$, $\theta_2 \in C(B)$. Therefore $a_1 \theta_1 a_2$ and $b_1 \theta_2 b_2$. Hence $\theta = \theta_1 \times \theta_2 \supseteq \theta(a_1, a_2) \times \theta(b_1, b_2)$. This proves (6).

(6) $\rightarrow$ (7): We have $((a_1, b), (a_2, b)) \in \theta(a_1, a_2) \times \theta(b_1, b_2) = \theta((a_1, b_1), (a_2, b_2))$ and similarly $((a, b_1), (a, b_2)) \in \theta((a_1, b_1), (a_2, b_2))$.

(7) $\rightarrow$ (5): We show that (3) holds for every $\theta \in C(A \times B)$. Suppose $(a_1, b_1) \theta (a_2, b_2)$. Then $\theta \supseteq \theta((a_1, b_1), (a_2, b_2))$. Therefore by (7), (3) follows.

THEOREM 4. *Let K be any class of similar algebras. Then K has property P if and only if*

(8) *for all $A, B \in K$, all $a_1, a_2 \in A$, and all $b_1, b_2, b \in B$,*

$$((a_1, b), (a_2, b)) \in \theta((a_1, b_1), (a_2, b_2)).$$

PROOF. The necessity of (8) is obvious by (7). To prove the sufficiency, we show that (7) holds for all $A, B \in K$. Let $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$. Then by (8)

$$((a_1, b), (a_2, b)) \in \theta((a_1, b_1), (a_2, b_2)).$$

Applying (8) to $B \times A$, we have

$$((b_1, a), (b_2, a)) \in \theta((b_1, a_1), (b_2, a_2)).$$

Using the canonical isomorphism of $B \times A$ with $A \times B$, we have

$$((a, b_1), (a, b_2)) \in \theta((a_1, b_1), (a_2, b_2)).$$

COROLLARY 2. *If K is an equational class such that for every $A \in K$ with two generators and for every $B \in K$ with three generators, $A \times B$ has property P , then K has property P .*

LEMMA 2. *If $u, v, c_0, c_1 \in A$, then $(u, v) \in \theta(c_0, c_1)$ if and only if for some $m \geq 1, n \geq 1$, there exist $(m+1)$ -ary polynomials $p_1, \dots, p_n$, elements z_{ij} of A for $1 \leq i \leq n, 1 \leq j \leq m$, and integers $k(1), \dots, k(n)$ such that*

$$\begin{aligned} k(i) &= 0 \quad \text{or} \quad 1 \quad \text{for } 1 \leq i \leq n, \\ u &= p_1(c_{k(1)}, z_{11}, \dots, z_{1m}), \quad v = p_n(c_{1-k(n)}, z_{n1}, \dots, z_{nm}), \\ p_i(c_{1-k(i)}, z_{i1}, \dots, z_{im}) &= p_{i+1}(c_{k(i+1)}, z_{i+1,1}, \dots, z_{i+1,m}) \\ &\quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

PROOF. This is a paraphrase of Theorem 3, p. 54 of [1]. As pointed out by G. Grätzer, we may take $m=n$ and $k(i)=0$ for even i and $k(i)=1$ for odd i .

THEOREM 5. *Let K be an equational class of algebras. Then K has property P if and only if for some $n \geq 1$ and some $m \geq 1$, there exist $(m+1)$ -ary polynomials $p_1, \dots, p_n$, binary polynomials $q_{ij}(x_0, x_1)$ and ternary polynomials $r_{ij}(x_0, x_1, x_2)$ for $1 \leq i \leq n, 1 \leq j \leq m$, and integers $k(1), \dots, k(n)$ which are 0 or 1 such that the following identities hold in all members of K :*

$$\begin{aligned} x_0 &= p_1(x_{k(1)}, q_{11}, \dots, q_{1m}), \quad x_1 = p_n(x_{1-k(n)}, q_{n1}, \dots, q_{nm}), \\ p_i(x_{1-k(i)}, q_{i1}, \dots, q_{im}) &= p_{i+1}(x_{k(i+1)}, q_{i+1,1}, \dots, q_{i+1,m}), \\ &\quad 1 \leq i \leq n-1, \\ x_2 &= p_1(x_{k(1)}, r_{11}, \dots, r_{1m}) = p_n(x_{1-k(n)}, r_{n1}, \dots, r_{nm}), \\ p_i(x_{1-k(i)}, r_{i1}, \dots, r_{im}) &= p_{i+1}(x_{k(i+1)}, r_{i+1,1}, \dots, r_{i+1,m}), \\ &\quad 1 \leq i \leq n-1. \end{aligned} \tag{9}$$

PROOF. Suppose K has property P . Let A be the free K algebra with two free generators x_0, x_1 and B be the free K algebra with three free generators x_0, x_1, x_2 . Then by Theorem 4, $((x_0, x_2), (x_1, x_2))$

$\in \theta((x_0, x_0), (x_1, x_1))$. To obtain (9), we use Lemma 2, the fact that every element of $A \times B$ is of the form $(q(x_0, x_1), r(x_0, x_1, x_2))$, and that $p((u_0, v_0), \dots, (u_m, v_m)) = (p(u_0, \dots, u_m), p(v_0, \dots, v_m))$ for every $(m+1)$ -ary polynomial. Conversely, if we assume (9), then by substituting $x_0 = a_0, x_1 = a_1$ in the first three lines of (9) and $x_0 = b_0, x_1 = b_1$ and $x_2 = b$ in the last two lines of (9), we see by Lemma 2 that (8) holds.

We close with some examples and remarks. As pointed out before, the class K_R of all rings with unit element has property P . More generally, if K is such that there exist binary polynomials $+$ and $\cdot$, and constants (or polynomials which are constant in K) 0 and 1 such that the identities $x \cdot 1 = x + 0 = 0 + x = x$ and $x \cdot 0 = 0$ hold in K , then K has property P . This follows from Theorem 5 with $n=1, m=2, p_1(x, y, z) = x \cdot y + z, q_{11} = 1, q_{12} = 0, r_{11} = 0, r_{12} = x_2$ and $k(1) = 0$.

Another example is the class K_L of all lattices. In this case, the condition of Theorem 5 holds with $n=1, m=2, p_1(x, y, z) = (x \wedge y) \vee z, q_{11} = x_0 \vee x_1, q_{12} = x_0 \wedge x_1, r_{11} = x_2, r_{12} = x_2$ and $k(1) = 0$. The fact that K_L has property P also follows from Corollary 1. Property P also extends to any class K (such as the class of lattice ordered groups) of algebras each of which is a lattice under some of its operations. There exist equational classes K having property P but such that $C(A \times B)$ is not distributive for some $A, B \in K$. As an example, let $K = K_R$ and $A = B =$ the ring of all polynomials in x, y over the rationals. If (p) denotes the principal ideal generated by p , then in $C(A)$, we have

$$(x + y) \cap ((x) \vee (y)) \not\subseteq ((x + y) \cap (x)) \vee ((x + y) \cap (y)).$$

We may define property P in the obvious way for arbitrary direct products. However, the following theorem shows that property P can only hold for essentially finite direct products.

THEOREM 6. *If $A = \prod_{i \in I} A_i$ has property P , then A_i has one element for all but a finite number of i .*

PROOF. Let $\theta = \{(x, y) : x(i) = y(i) \text{ for almost all } i\}$. Then $\theta \in C(A)$. If $\theta = \prod_{i \in I} \theta_i$, then θ_i must be U_{A_i} for all i , since $x\theta y$ whenever $x(j) = y(j)$ for all $j \in I, j \neq i$. Therefore $\theta = U_A$, from which it follows that A_i is trivial for almost all i .

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UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024