ON THE STABILITY OF THE FUNCTIONAL EQUATION

$$\varphi[f(x)] = g(x)\varphi(x) + F(x)$$

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ABSTRACT. This paper gives results on the stability of the solutions of $\varphi[f(x)] = g(x)\varphi(x) + F(x)$ where f , g and F are given functions and φ is unknown.

The problem of the stability of the functional equation in a narrow sense (i.e. not including differential and integral equations) was first discussed by D. H. Hyers [1], who has proved the stability of the Cauchy functional equation. Some results concerning the equation of isometry have been presented by D. H. Hyers and S. M. Ulam ([2], [3]).

In this note we discuss the stability of the linear functional equation

$$(1) \quad \varphi[f(x)] = g(x)\varphi(x) + F(x),$$

where f , g and F are given functions and φ is an unknown function. We shall assume the

HYPOTHESIS H_1 . The functions g and F are defined and continuous in an interval I and

$$(2) \quad g(x) \neq 0 \quad \text{for } x \in I.$$

HYPOTHESIS H_2 . The function f is defined, strictly increasing and continuous in I , moreover there exists a point $\xi \in I$ such that

$$\begin{aligned} [f(x) - x](\xi - x) &> 0 & \text{for } x \in I, \quad x \neq \xi, \\ [f(x) - \xi](\xi - x) &< 0 & \text{for } x \in I, \quad x \neq \xi. \end{aligned}$$

REMARK 1. It is easy to see that Hypothesis H_2 implies that $f(I) \subset I$ and that the point ξ is an attractive fixed point of f in the interval I .

REMARK 2. The interval I can be infinite. In such a case we require, instead of continuity in $-\infty$ (or in ∞), the existence of a finite limit of the solutions under discussion in $-\infty$ (or in ∞).

The equation (1) is stable in I in the sense of D. H. Hyers if there exists a positive number K such that for each $\epsilon > 0$ and for each solution ψ of the inequality

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$$(3) \quad |\psi[f(x)] - g(x)\psi(x) - F(x)| \leq \epsilon \quad \text{for } x \in I$$

there exists a solution φ of the equation (1) such that

$$(4) \quad |\psi(x) - \varphi(x)| \leq K\epsilon \quad \text{for } x \in I.$$

The equation (1) is not stable in the above sense even if we consider the continuous solutions of (1) and (3) only. In fact, consider the equation

$$(5) \quad \varphi(x+c) = \varphi(x),$$

where $I = (-\infty, \infty)$, c is a positive constant. In this case $\xi = \infty$. The solutions of the equation (5) are all the periodic functions with the period c . The only solutions of the equation (5) continuous at ∞ (i.e. possessing a finite limit at ∞), are the constant functions. But the inequality (3) has, for each positive ϵ , an unbounded continuous solution $\psi(x) = \epsilon x/c$. Therefore there does not exist any continuous solution of (1) satisfying (4) for a $K > 0$.

In such a situation we are going to modify the definition of stability, making use of the fact that the equation (1) is, for each positive integer n , equivalent to the equation

$$(6) \quad \varphi[f^n(x)] = G_n(x)\varphi(x) + G_n(x) \sum_{i=0}^{n-1} \frac{F[f^i(x)]}{G_{i+1}(x)},$$

where

$$(7) \quad G_n(x) = \prod_{i=0}^{n-1} g f^i(x) \quad \text{for } n = 1, 2, \dots, x \in I$$

and $f^i(x)$ is an i th iterate of the function f , i.e. $f^0(x) = x$, $f^{n+1}(x) = f[f^n(x)]$ for $n = 0, 1, \dots$.

DEFINITION. The equation (1) is called stable in the interval I with respect to the family $C(I)$ of all functions continuous in I , if there exists such a constant $K > 0$, that for each positive number ϵ and each continuous function ψ satisfying the inequality

$$(8) \quad \left| \psi[f^n(x)] - G_n(x)\psi(x) - G_n(x) \sum_{i=0}^{n-1} \frac{F[f^i(x)]}{G_{i+1}(x)} \right| \leq \epsilon$$

for $n = 1, 2, \dots, x \in I$,

there exists a solution φ of the equation (1), continuous in I , such that the inequality (4) holds.

Let us notice that the constant K depends only on the equation (i.e. it depends on the given functions f , g and F) and on the interval

I . It does not depend on the function ψ and on the number ϵ .

Although the equation (6) is equivalent to the equation (1), the inequality (8) is not equivalent to the inequality (3), as we shall see by virtue of the following results.

The number of the continuous solutions of the equation (1) depends on the behaviour of the sequence G_n . There are three cases possible:

(i) The limit

$$(9) \quad G(x) = \lim_{n \rightarrow \infty} G_n(x)$$

exists for all $x \in I$, $G(x)$ is continuous in I and

$$(10) \quad G(x) \neq 0 \quad \text{for } x \in I.$$

(ii) There exists an interval $J \subset I$, such that $G_n(x)$ tends to zero uniformly in J .

(iii) Neither of the cases (i) and (ii) occurs.

It is known (see [4]) that if the equation (1) has a continuous solution in I , then: in case (i) all the continuous solutions, are given by

$$(11) \quad \varphi(x) = \eta/G(x) + \varphi_0(x),$$

where $\eta = \varphi(\xi)$ is an arbitrary constant and $\varphi_0(x)$ is an arbitrary continuous solution of the equation (1); in case (ii) the equation (1) has a continuous solution depending on an arbitrary function (i.e. the solution can be chosen arbitrarily on an interval contained in I); in case (iii) the solution is unique.

THEOREM 1. *If the equation (1) has a continuous solution, if case (i) occurs and if*

$$(12) \quad \sup_{x \in I} \frac{1}{|G(x)|} < \infty,$$

then the equation (1) is stable in I with respect to $C(I)$.

PROOF. Let ϵ be an arbitrary positive number and ψ a continuous solution of the inequality (8). We obtain from (8)

$$\left| G_n(x) \right| \left| \frac{\psi[f^n(x)]}{G_n(x)} - \psi(x) - \sum_{i=0}^{n-1} \frac{F[f^i(x)]}{G_{i+1}(x)} \right| \leq \epsilon$$

for $n = 1, 2, \dots, x \in I$,

whence

$$(13) \quad \left| \frac{\psi[f^n(x)]}{G_n(x)} - \psi(x) - \sum_{i=0}^{n-1} \frac{F[f^i(x)]}{G_{i+1}(x)} \right| \leq \frac{\epsilon}{|G_n(x)|}$$

for $n = 1, 2, \dots, x \in I$.

As the equation (1) has a continuous solution in I , the series

$$(14) \quad - \sum_{n=0}^{\infty} \frac{F[f^n(x)]}{G_{n+1}(x)} = \varphi_0(x)$$

converges in I and φ_0 is a continuous solution of (1) in I (see [4]). If n tends to infinity, then $f^n(x) \rightarrow \xi$ for $x \in I$ (see [4]) and we obtain from (13)

$$(15) \quad |\psi(\xi)/G(x) - \psi(x) + \varphi_0(x)| \leq K\epsilon \quad \text{for } x \in I,$$

where

$$(16) \quad K = \sup_{x \in I} \frac{1}{|G(x)|}.$$

Putting

$$(17) \quad \varphi(x) = \varphi(\xi)/G(x) + \varphi_0(x) \quad \text{for } x \in I$$

we can see, by virtue of (11), that φ is a continuous solution of the equation (1) and we conclude from (15) that φ satisfies the inequality (4).

As the homogeneous equation

$$(18) \quad \varphi[f(x)] = g(x)\varphi(x)$$

always has a continuous solution, we obtain the following.

COROLLARY 1. *Under the hypothesis of Theorem 1 the equation (18) is stable.*

THEOREM 2. *Let the case (ii) occur. If there exists a point $x_0 \in I$ such that the interval $I_0 = [f(x_0), x_0]$ (or $I_0 = [x_0, f(x_0)]$) is contained in J , then equation (1) is stable in the interval $[\xi, x_0]$ (or in the interval $[x_0, \xi]$), provided that (1) has a continuous solution in I .*

PROOF. Without loss of generality we can assume that ξ is the left end of the interval I , i.e. $I_0 = [f(x_0), x_0]$. Let ψ be a continuous solution of the inequality (8) in I for a given positive number ϵ . Put

$$(21) \quad y_0 = \psi(x_0).$$

Let φ_0 be a function defined and continuous in the interval I_0 , such that

$$(22) \quad \varphi_0(x_0) = y_0$$

and

$$(23) \quad \varphi_0[f(x_0)] = g(x_0)\varphi_0(x_0) + F(x_0).$$

Now we have as a consequence of (8) and (23):

$$\begin{aligned} |\varphi_0[f(x_0)] - \psi[f(x_0)]| &= |g(x_0)\varphi_0(x_0) + F(x_0) - \psi_0[f(x_0)]| \\ &= |g(x_0)\psi(x_0) + F(x_0) - \psi[f(x_0)]| \leq \epsilon. \end{aligned}$$

So we can find a function φ_0 , continuous in I_0 , such that the inequality

$$(24) \quad |\varphi_0(x) - \psi(x)| \leq \epsilon \quad \text{for } x \in I_0$$

is satisfied. As it is known (see [4]), for such a function φ_0 there exists the unique continuous solution φ of the equation (1) in I , such that

$$(25) \quad \varphi(x) = \varphi_0(x) \quad \text{for } x \in I_0.$$

Let x be an arbitrary point of the interval $(\xi, f(x_0))$. There exists such a point $t \in I_0$ and a positive integer k , such that

$$(26) \quad x = f^k(t)$$

(see [4]). We obtain from (6), (24), (25) and (26)

$$\begin{aligned} |\psi(x) - \varphi(x)| &= |\psi[f^k(t)] - \varphi[f^k(t)]| \\ &= \left| \psi[f^k(t)] - G_k(t)\varphi(t) + G_k(t) \sum_{i=0}^{k-1} \frac{F[f^i(t)]}{G_{i+1}(t)} \right| \\ &\leq \left| \psi[f^k(t)] - G_k(t)\psi(t) + G_k(t) \sum_{i=0}^{k-1} \frac{F[f^i(t)]}{G_{i+1}(t)} \right| \\ &\quad + |G_k(t)| |\psi(t) - \varphi(t)| \leq \epsilon(1 + M), \end{aligned}$$

where

$$(27) \quad M = \sup_k \sup_{t \in I_0} |G_k(t)|.$$

M is a finite number, because the sequence G_n converges uniformly in the interval I_0 , by virtue of the condition $I_0 \subset J$. Thus the inequality (4) is fulfilled for all $x \in (\xi, f(x_0))$ with $K = 1 + M$. Because of the continuity of the functions φ and ψ in the point ξ it is also fulfilled at ξ . In view of (24) we obtain that the inequality (4) holds in $[\xi, x_0]$.

COROLLARY 2. *Since the hypothesis of Theorem 2 is fulfilled, the equation (18) is stable.*

The problem of the stability of the equation (1) is still open. Also, we do not know whether the condition (12) is essential.

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