SEMINORMAL OPERATORS WITH COMPACT SELF-COMMUTATORS

KEVIN CLANCEY

ABSTRACT. Putnam's inequality $\pi \|T^*T - TT^*\| \leq \text{meas}_2(\text{sp}(T))$ is verified for seminormal operators T when $T^*T - TT^*$ is compact. The spectrum of the seminormal singular integral operator defined for $f \in L^2(0,1)$ by

$$Tf(s) = sf(s) + i \left(\frac{\phi(s)}{\pi i} \int_0^1 \frac{f(t)\bar{\phi}(t)}{s-t} dt \right), \quad s \in [0, 1],$$

where ϕ is any fixed essentially bounded measurable function on $[0,1]$, is computed.

1. An operator A on a Hilbert space $\mathcal{H}$ is called seminormal in case its selfadjoint self-commutator $D = A^*A - AA^*$ is positive semi-definite ($D \geq 0$) or negative semidefinite ($D \leq 0$). When $D \geq 0$ the operator A is said to be hyponormal. Putnam [7] has conjectured that if A is seminormal then

$$(1) \quad \pi \|D\| \leq \text{meas}_2(\text{sp}(A))$$

where meas_2 denotes Lebesgue planar measure and $\text{sp}(A)$ denotes the spectrum of A . After this paper had been submitted for publication Putnam [6] discovered a proof of his remarkable inequality (1) for the general seminormal operator A . In §4 we will establish (1) in the case where D is compact. We will also determine the spectrum of the seminormal singular integral operator defined on $L^2(0, 1)$ by

$$(2) \quad Tf(s) = sf(s) + i \left(\frac{\phi(s)}{\pi i} \int_0^1 \frac{f(t)\bar{\phi}(t)}{s-t} dt \right) \quad \text{a.e. } s \in [0, 1],$$

where $f \in L^2(0, 1)$ and ϕ is an $L^\infty(0, 1)$ function. The singular integral in (2) is to be considered as a Cauchy principal value. The operator T is hyponormal, indeed if $(\cdot, \cdot)$ is the inner product in $L^2(0, 1)$ then for $f \in L^2(0, 1)$

$$(3) \quad (T^*T - TT^*)f = \frac{2}{\pi} (f, \phi)\phi.$$

We introduce for $0 \leq x \leq 1$ the notations

Received by the editors January 22, 1970.

AMS 1969 subject classifications. Primary 4615, 4710.

Key words and phrases. Seminormal operator, hyponormal operator, subnormal operator, singular integral.

$$\begin{aligned}
 M_x^\pm &= \operatorname{ess\,lim\,sup}_{t \rightarrow x} [\pm |\phi(t)|^2] \\
 (4) \qquad &= \lim_{n \rightarrow \infty} \operatorname{ess\,sup} \left[\pm |\phi(t)|^2 : t \in \left[x - \frac{1}{n}, x + \frac{1}{n} \right] \cap [0, 1] \right].
 \end{aligned}$$

In §3 we will prove

THEOREM 1. *If T is the operator defined by (2) then*

$$\operatorname{sp}(T) = \{x + iy : 0 \leq x \leq 1, M_x^- \leq y \leq M_x^+\}.$$

In [7] Putnam obtains Theorem 1 in the case where ϕ is Riemann integrable. Theorem 1 will yield (1) for all operators of the form (2). As is shown by an example in §3 the inequality may be strict. The essential spectrum of operators of the form (2) when ϕ is smooth has been studied by Schwartz [9].

2. For an operator B on $\mathcal{H}$ and complex number $z = x + iy$ we will write $B_z = B - zI$ and $B_z^* = B^* - \bar{z}I$. If $B = H + iJ$ is the cartesian decomposition of B where H and J are selfadjoint then $B_z = H_z + iJ_y$. The fine structure of the spectrum of an operator B will be needed; we will denote the compression spectrum of B by $\Sigma(B) = \{z : B_z^*x = 0, x \in \mathcal{H}, x \neq 0\}$, the approximate point spectrum by $\pi(B) = \{z : B_zx_n \rightarrow 0, x_n \in \mathcal{H}, \|x_n\| = 1\}$, and the point spectrum of B by $\pi_0(B) = \{z : B_zx = 0, x \in \mathcal{H}, x \neq 0\}$. Then clearly $\operatorname{sp}(B) = \Sigma(B) \cup \pi(B)$ and $\Sigma(B) = \{z : \bar{z} \in \pi_0(B^*)\}$. Moreover, it is well known that $\Sigma(B) \setminus \pi(B)$ is open (see, e.g. Halmos [3, p. 39]). In case H is selfadjoint we will write for β a Borel set $E(\beta)$ as the spectral projector of H associated with β . If $A = H + iJ$ is seminormal on $\mathcal{H}$ then $A_\beta = E(\beta)AE(\beta)$ is seminormal on $\mathcal{H}_\beta = E(\beta)\mathcal{H}$. It will be more convenient to write $A_\beta = \beta A \beta$ when there is only one spectral measure involved.

The following theorem is proved in Putnam [4, p. 46]:

THEOREM 2. *Let $A = H + iJ$ be seminormal then $\operatorname{pr}_x(\operatorname{sp}(A)) = \operatorname{sp}(H)$ and $\operatorname{pr}_y(\operatorname{sp}(A)) = \operatorname{sp}(J)$. (pr_x and pr_y denote the projections on the x and y axes respectively.)*

We will also use the following lemma of Putnam [7]:

LEMMA 1. *Let $A = H + iJ$ be hyponormal and $\Delta = [a, b]$, then $x + iy \in \pi(A)$ for $x \in (a, b)$ if and only if $x + iy \in \pi(A_\Delta)$.*

A selfadjoint operator H with spectral measure $E(\beta)$ for β a Borel set is said to be absolutely continuous in case the measure $\mu_x(\beta) = \|E(\beta)x\|^2$ is absolutely continuous with respect to Lebesgue measure for each $x \in \mathcal{H}$.

We now prove the following lemma inspired by Lemma 4 of Putnam [7].

LEMMA 2. Let $A = H + iJ$ be hyponormal with compact self-commutator and H absolutely continuous. If $\Delta = [a, b]$ and $z = x + iy \in \Sigma(A_\Delta) \setminus \pi(A_\Delta)$ for $x \in (a, b)$ then $z \in \Sigma(A) \setminus \pi(A)$.

PROOF. Set $c = \inf \{s : z \in \Sigma([s, b]A[s, b]), s \leq a\}$. Clearly $c \leq a$, if now c is finite we will first show $z \in \Sigma([c, b]A[c, b])$ so that the inf is a min. Select s_n decreasing to c with $s_n \leq a$ and $z \in \Sigma([s_n, b]A[s_n, b])$. Then there are $x_n \in \mathcal{H}_{[s_n, b]}$ with $\|x_n\| = 1$ and

$$(5) \quad [s_n, b]A_z^*[s_n, b]x_n = [s_n, b]A_z^*x_n = 0.$$

Select from the $x_n \in \mathcal{H}$ a weakly convergent subsequence $x_n \xrightarrow{w} q$. Now

$$(6) \quad \|[s_n, b]A_z[s_n, b]x_n\|^2 = (Dx_n, x_n)$$

so that if $q = 0$ then $Dx_n \rightarrow 0$ and then $[s_n, b]A_z[s_n, b]x_n$ would go to zero. We note that since (5) holds then $i[s_n, b]J_y[s_n, b]x_n = H_x x_n$ and so $[s_n, b]A_z[s_n, b]x_n = 2H_x x_n$ and therefore $H_x x_n$ also goes to zero. It follows that $[s_n, a]x_n \rightarrow 0$, indeed

$$(7) \quad \|H_x x_n\|^2 = \int_{[s_n, a]} (t - x)^2 d\|E_t x_n\|^2 + \int_{[a, b]} (t - x)^2 d\|E_t x_n\|^2 \\ \geq \|a - x\|^2 \|[s_n, a]x_n\|^2.$$

But then it follows that $[a, b]A_z[a, b]x_n \rightarrow 0$ for

$$(8) \quad \|[s_n, b]A_z[s_n, b]x_n\| \geq \|[a, b]A_z[s_n, b]x_n\| \\ \geq \|[a, b]A_z[a, b]x_n\| - \|[a, b]A_z[s_n, a]x_n\|$$

where $[s_n, b]A_z[s_n, b]x_n \rightarrow 0$ and $[s_n, a]x_n \rightarrow 0$. Since $\limsup \|[a, b]x_n\| > 0$ and $z \notin \pi(A_\Delta)$ then q cannot be zero. Clearly $q \in \mathcal{H}_{[c, b]}$ and since the operator H is absolutely continuous then $[s_n, b]A_z[s_n, b]p \rightarrow [c, b]A_z[c, b]p$ for $p \in \mathcal{H}$. If now $p \in \mathcal{H}_{[c, b]}$ then

$$(9) \quad 0 = ([s_n, b]A_z^*[s_n, b]x_n, p) = (x_n, [s_n, b]A_z[s_n, b]p) \rightarrow (q, [c, b]A_z[c, b]p).$$

Consequently $[c, b]A_z^*[c, b]q = 0$ so $z \in \Sigma([c, b]A[c, b])$. On the other hand if c is finite we may choose s_n increasing to c with $z \notin \text{sp}([s_n, b]A[s_n, b])$, certainly for $s_n < c$ we have $z \in \Sigma([s_n, b]A[s_n, b])$ and by Lemma 1 $z \notin \pi([s_n, b]A[s_n, b])$. Choose $\alpha > 0$ so that the circle centered at z of radius α is contained in the complement of $\text{sp}(A)$ (if no such α exists then $z \in \Sigma(A) \setminus \pi(A)$ and we are done) and in the strip $(a, b)X(-\infty, +\infty)$. Since $\Sigma(B) \setminus \pi(B)$ is open for any operator B and $s_n < c$ then the interior of this circle is contained in the comple-

ment of $\text{sp}([s_n, b]A[s_n, b])$ for all n . Now for $p \in \mathcal{H}_{[s_n, b]}$ there results

$$(10) \quad \|[s_n, b]A_z^*p\| = \|[s_n, b]A_z^*p\| \geq \alpha \|p\|.$$

This follows since the resolvent T_z^{-1} of a hyponormal operator T is bounded in norm by $\text{dist}^{-1}(z, \text{sp}(T))$ where dist denotes distance. See, e.g. Stampfli [11]. Then for $p \in \mathcal{H}_{[c, b]}$ we have

$$(11) \quad \|[s_n, b]A_z^*p\| > \alpha \|p\|$$

and since $[s_n, b]A_z^*p \rightarrow [c, b]A_z^*p$ it follows that $z \notin \Sigma([c, b]A[c, b])$. The conclusion is that $c = -\infty$, but then if $m = \min \text{sp}(H)$ we have $z \in \Sigma([m, b]A[m, b])$. Similarly set

$$c' = \sup \{s: z \in \Sigma([m, s]A[m, s]), s < m\}$$

as above one concludes that $c' = +\infty$. Then if $M = \max \text{sp}(H)$ it follows that $z \in \Sigma([m, M]A[m, M]) = \Sigma(A)$ and this completes the proof.

A hyponormal operator A on $\mathcal{H}$ is said to have the normal reducing subspace $\mathfrak{M}$ in case $\mathfrak{M}$ reduces A and the restriction of A to $\mathfrak{M}$ is a normal operator. The following result is due to Stampfli [10] (see, also Clancey [1], Putnam [5], Yoshino [12]).

THEOREM 3. *Let A be hyponormal and $z_0 \in \text{sp}(A)$. Assume that in some neighborhood N of z_0 , $\text{sp}(A) \cap N$ is contained in a Jordan arc. Then $A = A_1 \oplus A_2$ with A_1 normal and $\text{sp}(A_1) = \text{closure} [\text{sp}(A) \cap N]$.*

LEMMA 3. *Let $A = H + iJ$ with H absolutely continuous. Then for no $z_0 \in \text{sp}(A)$ does there exist a neighborhood N of z_0 with $\text{sp}(A) \cap N$ contained in a vertical segment.*

PROOF. If such a point z_0 exists then by Theorem 3 there is a subspace $\mathfrak{M}$ of $\mathcal{H}$ reducing A on which A acts like a normal operator, and the spectrum of the restriction A_1 of A to $\mathfrak{M}$ is contained in a vertical segment. Then the real part H_1 of A_1 would have an eigenvalue and so would H contradicting the fact that H is absolutely continuous.

3. We turn now to a proof of Theorem 1.

Let T be given as in (2) with $\phi \in L^\infty(0, 1)$. The operator T has the cartesian decomposition $H + iJ$ where

$$(12) \quad Hf(s) = sf(s) \quad \text{and} \quad Jf(s) = \frac{\phi(s)}{\pi i} \int_0^1 \frac{f(t)\bar{\phi}(t)}{s-t} dt, \\ s \in [0, 1], f \in L^2(0, 1).$$

If now β is any Borel set then Rosenblum [8] (see also Putnam [4, p. 135]) has shown that $\text{sp}(J_\beta) = [M_\beta^-, M_\beta^+]$ where $M_\beta^\pm = \pm \text{ess sup} [|\phi(t)|^2: t \in \beta]$.

In case $z = x + iy \in \{s + it: 0 < s < 1, M_s^- < t < M_s^+\}$ it must be in $\text{sp}(T)$. Otherwise for some $\delta > 0$ the square $[x - \delta, x + \delta]X[y - \delta, y + \delta]$ is a subset of the complement of $\text{sp}(T)$. In this case consider $T' = [x - \delta/2, x + \delta/2]T[x - \delta/2, x + \delta/2]$ then at least a part of one of the segments $\{x \pm \delta/2\}X[y - \delta/2, y + \delta/2]$ is contained in $\text{sp}(T')$ or else $\text{pr}_y(\text{sp}(T'))$ does not cover $[M_\Delta^-, M_\Delta^+]$ where $\Delta = [x - \delta/2, x + \delta/2]$ contradicting Rosenblum's result. Since the real part of T' is absolutely continuous this is impossible (by Lemma 3) unless some portion of $(x - \delta/2, x + \delta/2)X(y - \delta/2, y + \delta/2)$ is in $\text{sp}(T')$. From Lemma 1 we conclude this portion of $\text{sp}(T')$ must be in $\Sigma(T') \setminus \pi(T')$ which is open. Consequently $z \in \Sigma(T')$ and since the commutator of T is compact it follows from Lemma 2 that $z \in \Sigma(T)$. Putnam [7] has shown that $\text{sp}(T)$ is a subset of $\{s + it: 0 \leq s \leq 1, M_s^- \leq t \leq M_s^+\}$ and therefore the proof of Theorem 1 is complete.

Note that we do not determine the fine structure of $\text{sp}(T)$.

As was remarked above this determination of $\text{sp}(T)$ yields the inequality (1) for all operators of the form (2) but as the following example shows the inequality may be strict. Consider a set G in $[0, 1]$ of measure $1/2$ with the property that G intersects every proper open subinterval of $(0, 1)$ and its complement in a set of positive measure. Put $\phi = \text{characteristic function of } G$, then $M_s^\pm = \pm 1$ for $s \in [0, 1]$. Since

$$\|D\| = (2/\pi) \int_0^1 |\phi(t)|^2 dt = 1/\pi,$$

then the inequality (1) is strict for this T .

4. We will begin with a technical result on compact sets in the plane. Let K be a compact subset of the plane, we assume that K is in the interior of some square I_0XI_0 where $I_0 = [a, b]$ is a bounded interval. Define for any real number x , $I_x = \{y: x + iy \in K\}$ and $m(x) = \text{meas}_1(I_x)$ where meas_1 denotes one-dimensional Lebesgue measure. For any interval Δ define $\Gamma(\Delta) = \Delta X(-\infty, +\infty)$ and $m(\Delta) = \text{meas}_1(\text{pr}_y[\Gamma(\Delta) \cap K])$.

LEMMA 4. Let Δ_n be a sequence of intervals with $\Delta_n \supseteq \Delta_{n+1}$, length of $\Delta_n \rightarrow 0$ and $x \in \Delta_n$ for all n , then $m(\Delta_n)$ decreases to $m(x)$.

PROOF. Let $y \in I_0 \setminus I_x$, $y \neq a$, $y \neq b$, then for some $\delta > 0$ the square $[x - \delta, x + \delta]X[y - \delta, y + \delta]$ is contained in $I_0XI_0 \setminus K$. For n big enough

so that $\Delta_n \subset (x - \delta, x + \delta)$ we have $y \notin \text{pr}_v(\Gamma(\Delta_n) \cap K)$. Define $G_n = \{y \in I_0 : y \notin \text{pr}_v(\Gamma(\Delta_n) \cap K)\}$. Clearly G_n increases and we have just shown that G_n increases to $I_0 \setminus I_x$. So $m(\Delta_n) = \text{meas}_1(I_0) - \text{meas}_1(G_n)$ decreases to $\text{meas}_1(I_x) = m(x)$. We will also use

LEMMA 5. *Let $A = H + iJ$ be hyponormal with compact self-commutator and H absolutely continuous. Let $\Delta = [a, b]$ be any closed interval, then $\text{meas}_1(\text{sp}(J_\Delta)) = \text{meas}_1(\text{pr}_v(\text{sp}(A_\Delta))) \leq \text{meas}_1[\text{pr}_v(\Gamma(\Delta) \cap \text{sp}(A))]$.*

PROOF. The first equality follows from Theorem 2. The inequality will certainly be established if we can show $\text{sp}(A_\Delta) \subseteq \Gamma(\Delta) \cap \text{sp}(A)$. Now by Lemmas 1 and 2 we have $\Gamma((a, b)) \cap \text{sp}(A_\Delta) \subset \Gamma((a, b)) \cap \text{sp}(A)$. Suppose now that $a + iy \in \text{sp}(A_\Delta)$, then there must be a sequence $z_n = x_n + iy_n \rightarrow a + iy$ with $x_n \in (a, b)$ and $z_n \in \text{sp}(A_\Delta)$ or else in some neighborhood of $a + iy$, $\text{sp}(A_\Delta)$ would be contained in a vertical segment contradicting Lemma 3. But $z_n \in \text{sp}(A)$ by Lemma 1 or Lemma 2 so that $a + iy \in \text{sp}(A)$. A similar argument works in case $b + iy$ is in $\text{sp}(A_\Delta)$ and this completes the proof. We can now prove

THEOREM 4. *Let $T = H + iJ$ be hyponormal with compact self-commutator and with H absolutely continuous. Then $\pi \|T^*T - TT^*\| \leq \text{meas}_2(\text{sp}(T))$.*

PROOF. We mimic Putnam's proof of Theorem 3.5.1 of [4, p. 49]. Let $\Delta = (a, b]$, then equation (3.5.7) in Putnam [4, p. 49] yields for $x \in \mathcal{H}$

$$(13) \quad (\pi)^{1/2} \|D^{1/2}E(\Delta)x\| \leq [|\Delta| \text{meas}_1(\text{sp}(J_\Delta))]^{1/2} \|E(\Delta)x\|.$$

Define $Q(\Delta) = \text{meas}_1(\text{pr}_v(\Gamma(\Delta) \cap \text{sp}(T)))$, then Lemma 5 yields $\text{meas}_1(\text{sp}(J_\Delta)) \leq Q(\Delta')$ where $\Delta' = [a, b]$. Therefore we have

$$(14) \quad (\pi)^{1/2} \|D^{1/2}E(\Delta)x\| \leq [|\Delta| Q(\Delta')]^{1/2} \|E(\Delta)x\|.$$

If now $(c, d] \supset \text{sp}(H)$ and $P: c = c_0 < c_1 < \dots < c_N = d$ is a partition of $(c, d]$ into $\Delta_k = (c_{k-1}, c_k]$, $k = 1, \dots, N$, then

$$(15) \quad (\pi)^{1/2} \|D^{1/2}x\| \leq \left\{ \sum_{k=1}^N |\Delta_k| Q(\Delta_k) \right\}^{1/2} \|x\|.$$

Now define $F(x) = Q(\Delta'_k)$ on Δ_k so that

$$(16) \quad \pi(Dx, x) \leq \left\{ \int_c^d F(x) dx \right\} \|x\|^2.$$

Select now a sequence of partitions P_n with P_{n+1} a refinement of P_n and so that the norm of P_n goes to zero. If $F_n(x)$ corresponds to P_n as $F(x)$ did to P then $F_n(x)$ decreases to $m(x)$ [for $K = \text{sp}(T)$] a.e. (the

a.e. takes care of the endpoints of P_n). By the Lebesgue Monotone Convergence Theorem it follows from (16) that

$$(17) \quad \pi(Dx, x) \leq \left\{ \int_c^d m(x) dx \right\} \|x\|^2 = \text{meas}_2(\text{sp}(T)) \|x\|^2.$$

The following theorem due to Putnam [4, p. 42] shows that absolute continuity of H can be attained.

THEOREM 5. *Let $T = H + iJ$ be seminormal on the Hilbert space $\mathcal{H}$. Let $\mathfrak{M} = \mathfrak{M}_T$ denote the smallest subspace of $\mathcal{H}$ containing the range of D and invariant under both T and T^* , then the restriction of T to $\mathfrak{M}$ has absolutely continuous real part, moreover, T on the orthogonal complement of $\mathfrak{M}$ is normal.*

Then Theorems 4 and 5 combine to give

THEOREM 6. *Let $T = H + iJ$ be seminormal with compact self-commutator, then $\pi\|T^*T - TT^*\| \leq \text{meas}_2(\text{sp}(T))$.*

Obviously it follows from Theorem 6 that if T is seminormal with compact self-commutator and $\text{meas}_2(\text{sp}(T)) = 0$ then T is normal.

There exists examples of seminormal operators with compact self-commutators whose spectra have holes. Let $\{g_i\}_{i=-\infty}^{+\infty}$ be an orthonormal basis for $\mathcal{H}$ and set $Tg_i = a_i g_{i+1}$, $i = 0, \pm 1, \pm 2, \dots$, where

$$0 < m = \lim_{i \rightarrow -\infty} |a_i| \leq \dots \leq |a_i| \leq |a_{i+1}| \\ \leq \dots \leq \lim_{i \rightarrow \infty} |a_i| = M < \infty.$$

Then T is hyponormal indeed if $D = T^*T - TT^*$ then $Dg_i = [|a_{i+1}|^2 - |a_i|^2]g_i$ which is clearly compact. Also $\text{sp}(T) = \{z : m \leq |z| \leq M\}$ and for $z \notin \text{sp}(T)$ the operator T_z^{-1} is hyponormal with compact self-commutator. It follows that (1) holds for the operators T_z^{-1} . Compare with the result in Clancey [1].

REFERENCES

1. K. Clancey, *Putnam's inequality for the annulus*, Bull. Amer. Math. Soc. **76** (1970), 93-95.
2. ———, *Spectral properties of semi-normal operators*, Ph. D. Thesis, Purdue University, Lafayette, Ind., 1969.
3. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967. MR **34** #8178.
4. C. R. Putnam, *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36, Springer-Verlag, New York, 1967. MR **36** #707.

5. ———, *Hyponormal operators and low density spectra*, Math. Z. **106** (1968), 167–169. MR **38** #564.
6. ———, *An inequality for the area of hyponormal spectra*, (to appear).
7. ———, *The spectra of semi-normal singular integral operators*, Canad. J. Math. (to appear).
8. M. Rosenblum, *A spectral theory for self-adjoint singular integral operators*, Amer. J. Math. **88** (1966), 314–328. MR **33** #6453.
9. J. Schwartz, *Some results on the spectra and spectral resolutions of a class of singular integral operators*, Comm. Pure Appl. Math. **15** (1962), 75–90. MR **29** #480.
10. J. G. Stampfli, *A local spectral theory for operators*, J. Functional Analysis **4** (1969), 1–10.
11. ———, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476. MR **30** #3375.
12. T. Yoshino, *Spectral resolution of a hyponormal operator with the spectrum on a curve*, Tôhoku Math. J. (2) **19** (1967), 86–97. MR **35** #7154.

CARLETON UNIVERSITY, OTTAWA, CANADA