ISOTOPY AND HOMEOMORPHISM

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ABSTRACT. If X and Y are isotopically equivalent topological spaces, they are not necessarily homeomorphic. If X and Y are compact without boundary manifolds or n -pure simplicial complexes, then isotopy equivalence implies topological equivalence. An example of compact manifolds with boundaries which are not homeomorphic but are isotopically equivalent is given.

It has been hoped that algebraic invariants associated with various deleted products would help in the topological classification of spaces. These invariants are no better than the isotopy type of a space. The purpose of this paper is to show some cases where isotopy equivalence implies topological equivalence.

Two embeddings f and g from X into Y are isotopically equivalent, or simply, isotopic, if there is a homotopy between f and g in the space of embeddings from X into Y . Two spaces X and Y are isotopically equivalent or isotopic if there is an embedding of X into Y and one of Y into X such that each composition is isotopic to the identity map.

THEOREM 1. *Isotopic compact manifolds without boundary are homeomorphic.*

PROOF. If f is an isotopy equivalence of the manifold M into the manifold N , then f restricted to any connected component of M is an isotopy equivalence of that component with the corresponding component of N . Hence we may suppose M and N are connected. If g is an isotopy inverse to f , then $\dim M = \dim f(M) \leq \dim N = \dim g(N) \leq \dim M$ so M and N have the same dimension. By the Invariance of Domain Theorem, f is an open map, so $f(M)$ is an open closed subset of N . Hence f is a homeomorphism of M with N .

It is easy to show that the half-open interval is isotopic to the closed interval. In a similar way, using Brown's Collaring Theorem, we can prove that every compact manifold with boundary is isotopic to its interior. Since there exist 7-dimensional manifolds which are not homeomorphic, but have homeomorphic interiors, isotopy equivalence of manifolds with boundary does not imply topological equivalence.

Received by the editors February 21, 1970.

AMS 1969 subject classifications. Primary 5525; Secondary 5560.

Key words and phrases. Isotopy equivalence, n -pure simplicial complex, stringless simplicial complex.

Question. 1. What is the highest dimension of compact manifolds with boundary for which isotopy equivalence implies topological equivalence?

Question 2. Does isotopy equivalence imply homeomorphism for open manifolds?

To determine a class of simplicial complexes on which isotopy equivalence implies topological equivalence, we need some definitions. A simplicial complex K is n -pure if each point belongs to an n -simplex and no simplex has dimension $\geq n+1$. The set of all points of K which belong to a $n-1$ simplex which is contained in only one n -simplex is denoted by $\text{bd } K$ and those points are the boundary points of K . The set of points of K which have neighborhoods homeomorphic to R^n is denoted by $\text{int } K$ and those points are the interior points of K . For convenience the points of $\text{int } K$ or $\text{bd } K$ are called good points, and those not in $\text{int } K$ or $\text{bd } K$ are called bad points. If the closure of each connected component of $\text{int } K$ is compact, then K is stringless. The real line is an example of a simplicial complex which is not stringless. We notice that the bad points of K are those points which have arbitrarily small neighborhoods not homeomorphic to euclidean space.

We now state the two main theorems.

THEOREM 2. *If f is an isotopy equivalence between stringless one-pure locally finite simplicial complexes K and L then K is homeomorphic to L , although not necessarily by f .*

THEOREM 3. *If f is an isotopy equivalence between boundaryless, stringless, n -pure simplicial complexes K and L , then f is a homeomorphism between K and L .*

Although not necessary for the proofs of the above theorems, the following lemma is interesting and informative.

LEMMA 1. *If K and L are n -pure simplicial complexes, K is stringless and $\text{bd } K$ is empty, then a topological embedding f of K into L maps K onto a subcomplex of L .*

PROOF. It suffices to show that if $f(K) \cap s$ is nonempty, where s is an open n -simplex to L , then $f(K)$ contains $\bar{s}$. In this case $f(K)$ is the union of closed n -simplices of L , so it must be a subcomplex of L .

Suppose $f(K) \cap s$ is nonempty where s is an open n -simplex of L . Since $\text{bd } K$ is empty and since bad points of K are those whose neighborhoods are not homeomorphic to a subset of R^n , $f(K) \cap s = f(\text{int } K) \cap s$. Since $\text{int } K = \bigcup K_i$ where K_i is a component of $\text{int } K$,

we have $f(K_i) \cap s$ is nonempty for some i . Also $f(\bar{K}_i)$ is compact since K is stringless. Let x belong to $f(\bar{K}_i) \cap s$ and y belong to $s - f(\bar{K}_i)$. Let p be the last point on the line from x to y which is contained in $f(\bar{K}_i)$. For some q in $\bar{K}_i$, $p = f(q)$. Since $\bar{K}_i - K_i$ consists of boundary points or bad points of K , q belongs to K_i . Therefore there is a neighborhood about q which, by the Invariance of Domain Theorem, is carried to a neighborhood of p contained in $f(\bar{K}_i)$. This contradicts the definition of p so there does not exist a y in $s - f(\bar{K}_i)$. Since $f(\bar{K}_i)$ is compact, $f(K)$ contains $\bar{s}$ whenever $s \cap f(K)$ is nonempty.

The next lemma, which generalizes, will be used in proving Theorem 2.

LEMMA 2. *If f is an isotopy equivalence of K with itself where K is 1-pure and if p is a vertex of K belonging to at least three edges of K then $f(p) = p$. Moreover $f_t(p) = p$ for all t where f_t is the isotopy between 1_K and f .*

PROOF. If $f(p) \neq p$, then there is a t such that $f_t(p)$ belongs to the interior of one of the edges hitting p . Since f_t is an embedding, there is a connected neighborhood N of p which maps into the interior of a one-simplex. N is not an interval but its homeomorphic image is a connected subset of the line. This contradiction proves that $f_t(p) = p$ for all t .

Now we will prove Theorem 2. For the purpose of the proof we call a vertex a "real vertex" if it has at least three edges impinging on it. If g is the isotopy inverse of f , then Lemma 2 states that real vertices do not move when mapped by $g \circ f$. The idea of the proof is to show L may be subdivided so that $f(K)$ is a subcomplex of L which contains all the real vertices of L . Since the image of a real vertex is a real vertex of L , we subdivide L by adding the images of the other vertices of K to the vertices of L . We may add finitely many vertices to any simplex of L , and since K is stringless, this is all that will be necessary. Now L is subdivided so that $f(K)$ is a subcomplex.

Define $\text{ord}_K v$ for a vertex v of K to be the number of edges of K with v as a face. If v is a real vertex of L , $g(v)$ is a real vertex of K and $f \circ g(v) = v$ by Lemma 1. Therefore $f(K)$ is a subcomplex of L which contains all the real vertices of L . Moreover, since

$$\text{ord}_K v \leq \text{ord}_L f(v) \leq \text{ord}_K g \circ f(v) = \text{ord}_K v$$

and all these numbers are finite since K is locally finite, we have $\text{ord}_K v = \text{ord}_L f(v)$. Since L is stringless, L equals $f(K)$ with finite strings of one-simplices attached to $f(K)$ at vertices whose order is one. No attached string forms a loop for then $H_1(L, f(K)) \neq 0$ but $H_1(L, f(K))$

$=0$ for $f(K)$ is a weak deformation retract of L . (It is easy to show that if f is an isotopy equivalence between X and Y then $f(X)$ is a weak deformation retract of Y .) A homeomorphism from L to $f(K)$ may easily be constructed.

This theorem does not generalize to dimension 2. Now we prove Theorem 3.

If g is the isotopy inverse of f , we will prove that $f \circ g$ is a homeomorphism of K onto K . Let h_t be the isotopy between 1_K and $g \circ f$ and let $\{K_i\}$ be the components of $\text{int } K$.

First we show that if h is any embedding of K into K then $h(K_i) \cap K_j$ is nonempty implies $h(K_i) \supset K_j$. Suppose y belongs to $h(K_i) \cap K_j$ and x belongs to $K_j - h(K_i)$. Since K_j is path-connected there is a path α from y to x contained in K_j . Since $\alpha(0) = y$ belongs to $h(\bar{K}_i)$ and $\bar{K}_i$ is compact, there is a last t , say t_0 , such that $\alpha[0, t_0] \subset h(\bar{K}_i)$. But for all t , $\alpha(t)$ is a good point so $\alpha[0, t_0] \subset h(K_i)$ or $h^{-1} \circ \alpha[0, t_0] \subset K_i$. Since K_i is open this contradicts the definition of t_0 , unless $t_0 = 1$. In this case $h(K_i) \supset K_j$.

Now we show $g \circ f$ is onto K . If K_i is any component of $\text{int } K$, then there is a last t , say t_0 , such that $h_t(\bar{K}_i) \supset \bar{K}_i$ for all $t \leq t_0$. To see that $h_{t_0}(\bar{K}_i) \supset K_i$ let t_n be a sequence approaching t_0 from below. If $k \in K_i$ then there are points k_n in K_i such that $h_{t_n}(k_n) = k$ for all n . By relabelling we may assume k_n approaches $\bar{k}$ belonging to $\bar{K}_i$. Then $h_{t_0}(\bar{k}) = k$ by the continuity of the isotopy so $h_{t_0}(\bar{K}_i) \supset \bar{K}_i$.

Suppose that $t_0 < 1$. Since bad points are mapped into bad points, $h_{t_0}(K_i) \supset K_i$. Let $x \in K_i$ and choose neighborhood a N of t_0 such that $f_t(x)$ belongs to K_i for all t in N . For any t in N greater than t_0 , we have $h_t(K_i) \cap K_i$ is nonempty so $h_t(K_i) \supset K_i$. This contradicts the assumption that $t_0 < 1$, so $t_0 = 1$ and $g \circ f = h_1$ is onto.

Question 3. If K and L are stringless n -pure simplicial complexes where L is a subcomplex of K , does $H_n(K, L) = 0 = H_{n-1}(K, L)$ imply K is homeomorphic to L ?

Question 4. Find algebraic invariants that classify spaces up to isotopy type.

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