THE IMAGE OF $\mathfrak{U}_*(X) \rightarrow \mathfrak{N}_*(X)$

GEORGE E. MITCHELL

In [3] Milnor characterized those bordism classes in the Thom bordism ring $\mathfrak{N}_*$ which contain complex manifolds.¹ This was done by showing that the image of the weakly complex bordism ring $\mathfrak{U}_*$ under the natural ring homomorphism $j: \mathfrak{U}_* \rightarrow \mathfrak{N}_*$ is precisely $(\mathfrak{N}_*)^2$. This paper is devoted to an extension of the above result. That is, the image of the natural homomorphism $j_*: \mathfrak{U}_*(X) \rightarrow \mathfrak{N}_*(X)$ is computed for certain spaces X .

Conner and Floyd [1, p. 45] defined the Whitney numbers of an element $[M^n, f]_2 \in \mathfrak{N}_n(X)$ as follows: Let σ_n denote the fundamental classes of M^n , $w_1, \dots, w_n$ denote the Stiefel-Whitney classes of M^n , and let $h^m \in H^m(X; Z_2)$. For every partition $i_1 + \dots + i_k = n - m$ the number defined by

$$\langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot f^*(h^m), \sigma_n \rangle \in Z_2$$

is called the Whitney number of $[M^n, f]_2$ associated with h^m . (Here $\langle, \rangle$ denotes the Kronecker product.) For X a CW-complex of finite type these Whitney numbers characterize the elements of $\mathfrak{N}_*(X)$.

For X a CW-complex of finite type let $\{c_{n,i}\}$ be an additive basis for $H_n(X; Z_2)$ for each n . Since the natural homomorphism $\mu: \mathfrak{N}_n(X) \rightarrow H_n(X; Z_2)$ defined by

$$\mu([M^n, f]_2) = f_*(\sigma_n)$$

is onto, we may select a singular manifold $f_i: M_i^n \rightarrow X$ with $f_{i*}(\sigma_n) = c_{n,i}$. Define an $\mathfrak{N}_*$ -homomorphism $h: H_*(X; Z_2) \otimes \mathfrak{N}_* \rightarrow \mathfrak{N}_*(X)$ by $h(c_{n,i} \otimes 1) = [M_i^n, f_i]_2$. For X a CW-complex of finite type h is an isomorphism. (See [1, (17.1)].)

Now let X be a CW-complex for which the $\mathfrak{U}_*$ -spectral sequence collapses. By [2, (1.5)] we have that the natural map $\mathfrak{U}_*(X) \rightarrow H_*(X; Z)$ is an epimorphism. Let $\beta_*: H_*(X; Z) \rightarrow H_*(X; Z_2)$ be the reduction mod 2 and let W be the image of β_* in $H_*(X; Z_2)$. Let $\{c_{q,i}\}$ be a homogeneous basis for W . (q denotes the dimension of $c_{q,i}$.) Extend this basis to a basis of $H_*(X; Z_2)$ by adding homogeneous basis elements $\{\tilde{c}_{q,j}\}$. All of the elements $c_{q,i}$ can be represented by weakly complex manifolds V_i^q and maps $f_i: V_i^q \rightarrow X$ such that

Received by the editors, May 25, 1969.

AMS subject classifications. Primary 5530, 5560, 5710.

¹ All manifolds are assumed to be smooth, compact and without boundary.

$\beta_*(f_*(\sigma'_{q,i})) = c_{q,i}$, where $\sigma'_{q,i} \in H_q(V_i^q; Z)$ is the orientation class of V_i^q . For the basis elements $\tilde{c}_{q,j}$ we pick (nonweakly complex) manifolds M_j^q and maps $f_j: M_j^q \rightarrow X$ such that $f_{j*}(\sigma_{q,j}) = c_{q,j}$, where $\sigma_{q,j} \in H_q(M_j^q; Z_2)$ is the fundamental class.

Let $\{c^{q,i}\} \cup \{\tilde{c}^{q,j}\} \subset H^*(X; Z_2)$ be a dual basis for $H^*(X; Z_2)$ such that

$$\begin{aligned}\langle c^{q,i}, c_{q,i} \rangle &= 1 = \langle \tilde{c}^{q,j}, \tilde{c}_{q,j} \rangle && \text{for all } i \text{ and } j, \\ \langle c^{q,i}, c_{q,i'} \rangle &= 0 = \langle \tilde{c}^{q,j}, \tilde{c}_{q,j'} \rangle && \text{for } i \neq i' \text{ and } j \neq j', \text{ and} \\ \langle \tilde{c}^{q,j}, c_{q,i} \rangle &= 0 = \langle c^{q,i}, \tilde{c}_{q,j} \rangle && \text{for all } i \text{ and } j.\end{aligned}$$

Let $W' \subset H^*(X; Z_2)$ be the subspace spanned by $\{\tilde{c}^{q,j}\}$. Note that W' consists of exactly those elements $x \in H^*(X; Z_2)$ for which $\langle x, \text{Im } \beta_* \rangle = 0$.

THEOREM 1. *Let X be a CW-complex of finite type for which the $\mathfrak{U}_*$ -spectral sequence collapses. If $[V^n, f]_2$ is the image of $[V^n, f]$ under the natural map $j_*: \mathfrak{U}_*(X) \rightarrow \mathfrak{H}_*(X)$, then all the Whitney numbers of $[V^n, f]_2$ involving an odd w_i (that is, i is odd) or a homogeneous element $x \in W'$ are zero.*

PROOF. The first part follows from the fact that the odd Stiefel-Whitney classes of a weakly complex manifold are all zero.

Suppose x is a homogeneous element of W' . Consider any Whitney number $\langle w_{2i_1} \cdots w_{2i_k} \cdot f^*(x), \sigma_n \rangle$. Here σ_n is the mod 2 reduction of the orientation class $\sigma'_n \in H_n(V^n; Z)$; that is, $\beta_*(\sigma'_n) = \sigma_n$. Since the even dimensional Stiefel-Whitney classes are the mod 2 reduction of the Chern classes [3] we may write

$$\begin{aligned}\langle w_{2i_1} \cdots w_{2i_k} \cdot f^*(x), \sigma_n \rangle &= \langle \beta^*(c_{i_1}) \cdots \beta^*(c_{i_k}) \cdot f^*(x), \beta_*(\sigma'_n) \rangle \\ &= \langle f^*(x), (\beta^*(c_{i_1}) \cdots \beta^*(c_{i_k})) \cap \beta_*(\sigma'_n) \rangle \\ &= \langle f^*(x), \beta_*((c_{i_1} \cdots c_{i_k}) \cap \sigma_n) \rangle \\ &= \langle x, f_*\beta_*((c_{i_1} \cdots c_{i_k}) \cap \sigma_n) \rangle \\ &= \langle x, \beta_*f_*((c_{i_1} \cdots c_{i_k}) \cap \sigma_n) \rangle.\end{aligned}$$

(Here $c_{i_1}, \dots, c_{i_k}$ are Chern classes of V^n .) The last term is zero, since $\langle x, \text{Im } \beta_* \rangle = 0$. The conclusion follows.

THEOREM 2. *Let X be a CW-complex of finite type. If all the Whitney numbers of $[M^n, f]_2$ involving an odd w_i or a homogeneous element $x \in W' \subset H^*(X; Z_2)$ are zero, then $[M^n, f]_2$ is in the image of $j_*: \mathfrak{U}_*(X) \rightarrow \mathfrak{H}_*(X)$.*

PROOF. By [1, (17.1)] we may write

$$[M^n, f]_2 = \sum_{q, i} [V_i^{n-q}, f_i]_2 \cdot [Q_i^q]_2 + \sum_{q, j} [M_j^{n-q}, f_j]_2 \cdot [B_j^q]_2.$$

We will show by induction on q that $[Q_i^q]_2$ contains a weakly complex manifold and that $[B_j^q]_2 = 0$. Suppose true for $q < q_0$, and consider the Whitney number

$$\langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot f^*(\tilde{c}^{n-q_0, j_0}), \sigma_n \rangle,$$

where $i_k + \dots + i_1 = q_0$ are picked arbitrarily. By assumption this number is zero. Now

$$\begin{aligned} \langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot f^*(\tilde{c}^{n-q_0, j_0}), \sigma_n \rangle \\ = \sum_{q, i} \langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot (f_i \pi)^*(\tilde{c}^{n-q_0, j_0}), \sigma_{n-q, i} \times \sigma_{q, i} \rangle \\ + \sum_{q, j} \langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot (f_j \pi)^*(\tilde{c}^{n-q_0, j_0}), \sigma_{n-q, j} \times \sigma_{q, j} \rangle. \end{aligned}$$

All terms for which $q > q_0$ are zero by dimensional considerations. Since all the terms involving $q < q_0$ are Whitney numbers of weakly complex manifolds (by the induction hypothesis), they are also all zero. Hence this sum reduces to

$$\begin{aligned} \sum_i \langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot (f_i \pi)^*(\tilde{c}^{n-q_0, j_0}), \sigma_{n-q_0, i} \times \sigma_{q_0, i} \rangle \\ + \sum_j \langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot (f_j \pi)^*(\tilde{c}^{n-q_0, j_0}), \sigma_{n-q_0, j} \times \sigma_{q_0, j} \rangle. \end{aligned}$$

All the terms of these sums are zero except for the term in which $j = j_0$. That is, the only nonzero term is

$$\langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot (f_{j_0} \pi)^*(\tilde{c}^{n-q_0, j_0}), \sigma_{n-q_0, j_0} \times \sigma_{q_0, j_0} \rangle.$$

But this term is equal to the product

$$\langle f_{j_0}^*(\tilde{c}^{n-q_0, j_0}), \sigma_{n-q_0, j_0} \rangle \cdot \langle v_{i_1} \cdot \dots \cdot v_{i_k}, \sigma_{q_0, j_0} \rangle = \langle v_{i_1} \cdot \dots \cdot v_{i_k}, \sigma_{q_0, j_0} \rangle,$$

where the v_i 's are the Stiefel-Whitney classes of $B_{j_0}^{q_0}$. But by assumption this Whitney number is zero for any partition $(i_1, \dots, i_k)$ with $i_1 + \dots + i_k = q_0$ (since $\tilde{c}^{n-q_0, j_0}$ is in W'). Hence $[B_{j_0}^{q_0}]_2 = 0$. Since X is of finite type, there are only a finite number of choices for j_0 . Hence the same argument shows that $[B_j^{q_0}]_2 = 0$ for all j .

Now consider the Whitney number

$$\langle w_{i_1} \cdot \dots \cdot w_{i_k} \cdot f^*(c^{n-q_0, i_0}), \sigma_n \rangle,$$

where at least one of the w_i 's has i odd. By assumption this number is zero. The above argument shows that

$$\langle w_{i_1} \cdots w_{i_k} \cdot f^*(c^{n-q_0, i_0}), \sigma_n \rangle = \langle v_{i_1} \cdots v_{i_k}, \sigma_{q_0, i_0} \rangle,$$

where the v_i 's are the Stiefel-Whitney classes of $Q_{i_0}^{q_0}$. Since this number is zero for any partition $(i_1, \dots, i_k)$ with $i_1 + \dots + i_k = q_0$ for which some i_r ($1 \leq r \leq k$) is odd, it follows from Milnor [3] that $[Q_{i_0}^{q_0}]_2$ contains a weakly complex manifold.

We have shown that we may write

$$[M^n, f]_2 = \sum_{q, i} [V_i^{n-q}, f]_2 \cdot [Q_i^q]_2,$$

where the V_i^{n-q} are weakly complex manifolds and the $[Q_i^q]_2$ contain weakly complex manifolds. It follows that $[M^n, f]_2$ is in the image of $j_*: \mathfrak{U}_*(X) \rightarrow \mathfrak{N}_*(X)$. This completes the proof.

In conclusion we have proven the following:

THEOREM. *For X a CW-complex of finite type for which the $\mathfrak{U}_*$ -spectral sequence collapses we have*

$$\text{Im } j_* = h(\text{Im } \beta_* \otimes (\mathfrak{N}_*)^2) \subset \mathfrak{N}_*(X),$$

where h is the isomorphism $H_*(X; \mathbb{Z}_2) \otimes \mathfrak{N}_* \rightarrow \mathfrak{N}_*(X)$.

REFERENCES

1. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 33, Academic Press, New York and Springer-Verlag, Berlin and New York, 1964. MR 31 #750.
2. ———, *Torsion in SU-bordism*, Mem. Amer. Math. Soc. No. 60 (1966). MR 32 #6471.
3. J. W. Milnor, *On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds*, Topology 3 (1965), 223–230. MR 31 #5207.
4. N. Steenrod, *The topology of fibre bundles*, Princeton Math. Series, vol. 14, Princeton Univ. Press, Princeton, N. J., 1951. MR 12, 522.

WEST VIRGINIA UNIVERSITY, MORGANTOWN, WEST VIRGINIA 26506