A SPECIAL CLASS OF MOUFANG LOOPS

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ABSTRACT. Loops satisfying the identical relation $(xy)(zx) = [x(yz)]x$ are known as Moufang loops. In the present paper considered is a class of loops satisfying the identical relation

$$(1) \quad (xy)(zx^\lambda) = [x(yz)]x^\lambda,$$

where x^λ is the image of x under some mapping λ of the loop into itself. Loops satisfying (1) are called M -loops.

If $\lambda: x \rightarrow x^k$, k an integer, (1) is called an M_k -law. It is shown that every M -loop is Moufang, and every $x^k = x^{-1} \cdot x^\lambda$ belongs to the nucleus.

Furthermore, if a loop satisfies (1) so do all its loop-isotopes.

I. Introduction. In a commutative loop the mapping $x \rightarrow x^3$ is a centralizing endomorphism. The question arose whether one could generalize this fact and determine a class of loops with a centralizing endomorphism $x \rightarrow x^n$ for an arbitrary integer n . The first idea that comes to mind is to ask, even in the noncommutative case, how Moufang loops generalize when the Moufang law $(xy)(zx) = (x(yz))x$ is replaced by a seemingly more general one

$$(1) \quad (xy)(zx^k) = (x(yz))x^k, \quad k > 1.$$

In this paper we answer this question obtaining a number of results, all containing as particular case, for $k=1$, corresponding ones for Moufang loops. In fact, we go even further, replacing the mapping $x \rightarrow x^n$ by any single-valued mapping λ of the loop into itself.¹

In a preliminary report (Notices Amer. Math. Soc. 16 (1969), Abstract 664-753) some of these results were stated as generalizations of corresponding ones for Moufang loops. Somewhat surprisingly, it turns out that if (1) holds for any integer k it also holds for $k=1$ (see Theorem 1). It follows that the structure induced on a loop by (1) for $k>1$ is not more general than that of Moufang loops, but, on the contrary, (1) defines a restriction of them.

II. Basic definitions and properties of M -loops. Let G be a loop and λ, δ two single-valued mappings of G into itself such that $\lambda = 1 + \delta$ in the sense that

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$$(2) \quad x^\lambda = x \cdot x^\delta \quad \text{for every } x \text{ in } G.$$

THEOREM 1. *If G satisfies the identical relation*

$$(3) \quad (xy)(zx^\lambda) = (x(yz))x^\lambda$$

then G is Moufang and $G\delta \subset N$, where N is the nucleus of G . Conversely, if G is Moufang with nucleus N , the set D of all maps $\delta: G \rightarrow N$ is an associative left neoring, and (3) is an identical relation in G for every λ defined by (2) for every $\delta \in D$.

To prove this theorem we will need two lemmas:

LEMMA 1. *If G satisfies (3) then $L(x)R(x^\lambda) = R(x^\lambda)L(x)$, where $L(x)$, $R(x)$ are left and right multiplications by x , respectively.*

The proof follows from (3) with $y = 1$.

LEMMA 2. *If a loop G satisfies (3) then G has the inverse property.*

PROOF. We define x^{-1} by $xx^{-1} = 1$. Using (3) with $y = x^{-1}$ we have $(x(x^{-1}z))x^\lambda = (xx^{-1})(zx^\lambda) = zx^\lambda$, or

$$(4) \quad x(x^{-1}z) = z,$$

which is the left inverse property. From (4) with $z = x: x(x^{-1}x) = x$, $x^{-1}x = 1$ and

$$(5) \quad (x^{-1})^{-1} = x.$$

Setting $z = y^{-1}x^{-1}$ in (3) we have $(xy)((y^{-1}x^{-1})x^\lambda) = (x(y(y^{-1}x^{-1})))x^\lambda = x^\lambda$ and

$$(6) \quad (xy)^{-1} = y^{-1}x^{-1}.$$

To prove the right inverse property we choose $y^{-1} = zx^\lambda$, $x^\lambda = z^{-1}y^{-1} = (yz)^{-1}$, and substitute it into (3):

$$(xy)y^{-1} = (xy)(zx^\lambda) = x((yz)x^\lambda) = x((yz)(yz)^{-1}) = x.$$

This completes the proof of Lemma 2.

We can now prove Theorem 1:

(i) An autotopism of a loop G is a triple (U, V, W) of permutations of L such that $yU \cdot zV = (yz)W$ for all $x, y, z \in G$. If (U, V, W) and (U', V', W') are autotopisms of G then the following triples are also autotopisms of G : (UU', VV', WW') , (W, JVJ, U) and (JUJ, W, V) , where J is the inverse mapping [1, p. 112]. If (3) holds in G , then the following triples are autotopisms:

$$A(x) = (L(x), R(x^\lambda), L(x)R(x^\lambda)), \quad B(x) = (R(x^{-1}), L(x)R(x^\lambda), R(x^\lambda)),$$

$$A(x)B(x)^{-1} = (L(x)R(x), L(x)^{-1}, L(x)), \quad C(x) = (L(x), R(x), L(x)R(x)).$$

Since $C(x)$ is an autotopism of G , G is Moufang.

(ii) Being Moufang, G is diassociative. Using diassociativity of G and (3) we have an autotopism $D(x) = A(x)C(x^{-1}) = (I, R(x^\delta), R(x^\delta))$, or $y \cdot zx^\delta = (yz)x^\delta$, which implies that $x^\delta \in N$ for all $x \in G$. This proves the first part of the theorem.

(iii) To prove the converse we consider a Moufang loop G and the set $D = \{\delta\}$ of all mappings of G into N . It is easy to show that D is an associative left neoring with respect to operations $(\oplus)$ and $(\circ)$ defined as follows:

$$x(\delta_1 \oplus \delta_2) = x\delta_1 \cdot x\delta_2, \quad x \in G,$$

$$x(\delta_1 \circ \delta_2) = (x\delta_1)\delta_2.$$

(A left neoring is a system $(R, +, \cdot)$ with two binary operations such that $(R, +)$ is a loop with identity 0, R is closed under $(\cdot)$, $x \cdot 0 = 0$ and $x(y+z) = xy + xz$ for all $x, y, z \in G$ [3, p. 50].) In our case $(D, \oplus)$ is a group, not necessarily Abelian, the identity being the trivial mapping $\delta_0: x \rightarrow e$. $(D, \circ)$ is closed and $\delta_i \circ \delta_0 = \delta_0$ for every δ_i . The left distributive law also holds. The right distributive law holds in $(D, \oplus, \circ)$ only if every $\delta \in D$ is an endomorphism.

(iv) Finally, from the Moufang law with $x^\delta \in N$, (3) follows:

$$(xy)(zx)x^\delta = ((x(yz))x)x^\delta, \quad (xy)(zx^\lambda) = (x(yz))x^\lambda.$$

For the sake of convenience we wish to introduce the following definitions. A loop G will be called an M -loop if it satisfies (3) for some single-valued mapping λ of G into itself. The identical relation (3) will be called the M_λ -law. Every Moufang loop is an M -loop since it satisfies (3) for λ the identical mapping. If a loop G satisfies the M_λ -law for λ being the trivial mapping $x \rightarrow e$, G is a group. Of a special interest are loops where λ is a power-mapping $x \rightarrow x^k$ for an integer k . The identical relation (3) will in this case be called an M_k -law. If G is a loop, and k is the smallest positive integer > 1 such that the M_k -law holds in G but $x^{k-1} = 1$ does not hold, the loop G will be called an M_k -loop. A loop which is an M -loop but not an M_k -loop will be called *strictly Moufang*.

The following properties of M -loops are corollaries of Theorem 1:

COROLLARY 1. *In an M_k -loop, the $x^{(n(k-1)+3)}$ is a companion of some pseudo-automorphism upon G , for every $x \in G$.*

This follows from Theorem 1 and Bruck's theorem that every element of NG^3 of a Moufang loop G is a companion [2, p. 59].

COROLLARY 2. *If a commutative loop G satisfies an M_k -law for $k \not\equiv 1 \pmod{3}$ then G is an Abelian group.*

PROOF. If G satisfies an M_k -law, then $x^{k-1} \in N$ for every $x \in G$. On the other hand, in a commutative Moufang loop $x^3 \in N$. If $(k-1, 3) = 1$, or $k \not\equiv 1 \pmod{3}$, every $x \in N$ and G is a group.

Two loops $(G, \cdot)$ and $(G, \circ)$ are called isotopic if there exists a triple of permutations (α, β, γ) of G such that $x\alpha \cdot y\beta = (x \circ y)\gamma$. If $\gamma = 1$, $(G, \oplus)$ is called a principal isotope of $(G, \cdot)$. A property p of a loop is called a universal property if it is invariant under loop-isotopy.

THEOREM 2. *An M_λ -law is a universal property of loops.*

PROOF. Let the loops $(G, \cdot)$ and $(G, \circ)$ be isotopic and $(G, \cdot)$ satisfy an M_λ -law. We have to show that the M_λ -law holds also in $(G, \circ)$. Since the Moufang property is universal for loops, $(G, \circ)$ is Moufang. Without loss of generality, we can assume that $(G, \circ)$ is a principal isotope of $(G, \cdot)$ and $x \circ y = xf \cdot f^{-1}y$ [1, p. 129]. If x^δ belongs to the nucleus of $(G, \cdot)$ it also belongs to the nucleus of $(G, \circ)$:

$$\begin{aligned}(y \circ z) \circ x^\delta &= ((yf \cdot f^{-1}z)f)(f^{-1}x^\delta) = (yf \cdot f^{-1}z)x^\delta, \\ y \circ (z \circ x^\delta) &= (yf)(f^{-1}(zf \cdot f^{-1}x^\delta)) = (yf)(f^{-1} \cdot zx^\delta) = (yf \cdot f^{-1}z)x^\delta,\end{aligned}$$

and $(y \circ z) \circ x^\delta = y \circ (z \circ x^\delta)$. Now, $x^\delta \in N(G, \circ)$ together with the Moufang law implies the M_λ -law for $(G, \circ)$.

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