IMBEDDING FREE CYCLIC GROUP ACTIONS IN CIRCLE GROUP ACTIONS

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ABSTRACT. Suppose a closed, orientable, irreducible 3-manifold M admits a free cyclic group action of prime order. We consider the problem of determining when M admits an effective action of the circle group $SO(2)$ in which the cyclic action is imbedded. The main result is that if the Z_k action is "Z-classified", then it is weakly equivalent to a Z_k action imbedded in an effective action of $SO(2)$ if and only if some homeomorphism generating the first Z_k action is homotopic to the identity homeomorphism on M .

Let Z_k be a cyclic group of prime order $k \geq 2$. A Z_k action on a 3-manifold M is said to be *proper* if some homeomorphism generating the action is homotopic to the identity on M . Let M^* denote the orbit space M/Z_k and $\pi: M \rightarrow M^*$ denote the projection. We shall say that a Z_k action on M is *Z-classified* if there exists a commutative diagram

$$\begin{array}{ccc} M & \rightarrow & S^1 \\ \downarrow \pi & & \downarrow p \\ M^* & \rightarrow & S^1 \end{array}$$

where $p: S^1 \rightarrow S^1$ is the usual k -sheeted covering of the circle. In particular, this is always the case when $H_1(M^*; \mathbb{Z})$ has no k -torsion [5].

A Z_k action on M is determined by a map $\mu: Z_k \times M \rightarrow M$, where $\mu(g)(x) = \mu(g, x)$ is a homeomorphism for each $g \in Z_k$ and $\mu(g \circ g') = \mu(g) \circ \mu(g')$. The Z_k actions determined by the maps $\mu, \nu: Z_k \times M \rightarrow M$ are said to be *weakly equivalent* if there is a group automorphism $A: Z_k \rightarrow Z_k$ and a homeomorphism $H: M \rightarrow M$ such that $\mu(g) = H^{-1} \circ \nu(A(g)) \circ H$ for all $g \in Z_k$.

THEOREM. *Let M be a closed, orientable, irreducible 3-manifold. A Z_k -classified free Z_k action on M ($k \geq 2$ and prime) is proper if and only if it is weakly equivalent to some Z_k action embedded in an effective $SO(2)$ action on M .*

PROOF. Let $p: S^1 \rightarrow S^1_0$ be the standard k -fold covering of the circle

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S_0^1 . There are maps f and $\tilde{f}$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & S^1 \\ \downarrow \pi & & \downarrow p \\ M^* & \xrightarrow{f} & S_0^1 \end{array}$$

commutes. By the homotopy covering property, for any map $M^* \rightarrow S_0^1$ homotopic to f (with the appropriate map $M \rightarrow S^1$) the diagram also commutes. Hence for some $a \in S_0^1$ we may assume that $F = f^{-1}(a)$ is a closed, connected, orientable, polyhedral, two-sided 2-manifold such that the homomorphism induced by inclusion $i_*: \pi_1(F) \rightarrow \pi_1(M^*)$ is one-to-one [7, Lemma 1].

Let $p^{-1}(a) = \{a_i\}_{i=1}^k \subset S^1$. By a suitable choice of labeling, we may let A be the arc in S^1 with endpoints a_1 and a_2 such that no other a_i lies on A . Let $U = \tilde{f}^{-1}(A \setminus \{a_1, a_2\})$ and $T = \tilde{f}^{-1}(a_1)$. There is a homeomorphism $h: M \rightarrow M$ generating the Z_k action such that the pair (U, T) forms an h -partition [5] with both U and T connected. This means that $M = \bigcup_{i=1}^k [h^i(U) \cup h^i(T)]$, $\text{Bd}(U) = T \cup h(T)$, and $h^i(T) \cap h^j(T) = \emptyset$ and $h^i(U) \cap h^j(U) = \emptyset$ if $i \not\equiv j \pmod{k}$.

By using the fact that $h \simeq 1_M$ we show in [5] that $\pi_1(M)$ is an extension of $\pi_1(T)$ by the infinite cyclic group Z . Now split M along T to obtain a manifold E with boundary two disjoint copies T_0, T_1 of T . The maps $\pi_1(T_0) \rightarrow \pi_1(E)$ and $\pi_1(T_1) \rightarrow \pi_1(E)$ are isomorphisms. Stallings shows in the proof of Theorem 2 of [4] that $E \approx T \times [0, 1]$. The homeomorphism $\psi: T \rightarrow T$ repairing this cut (i.e. $M \approx T \times R / \{(x, t) \sim (\psi(x), t+1)\} = T \times R / \psi$) can be chosen so that ψ is of finite order n , where $(n, k) = 1$ [6]. For convenience in notation, we identify M with $T \times R / \psi$, and denote points in M by $[x, t]$.

In E we have $h^i(T)$ separating T_0 from T_1 for $1 \leq i < k$. It follows from Theorem 7.2 of [1] that $h: M \rightarrow M$ is equivalent to a homeomorphism, again denoted by h , such that in E we have $h^i(T) = T \times \{i/k\}$ ($1 \leq i \leq k$). The new h may clearly be chosen so that in E we have $h^i: T \times \{\lambda\} \rightarrow T \times \{\lambda + i/k\}$ for all $\lambda \geq 0$ with $\lambda + i/k \leq 1$ ($i = 1, \dots, k-1$).

Let $\rho: T \times [0, 1] \rightarrow T$ be the projection to the first factor. Since $h^k = 1_M$, we may conclude that the composite

$$T = T \times \{0\} \xrightarrow{h^k} T \times \{1\} \subset T \times [0, 1] \xrightarrow{\rho} T$$

is equivalent to $\psi: T \rightarrow T$, the homeomorphism repairing the cut of M along T .

Let $h_1: T \rightarrow T$ be the homeomorphism defined by the composite

$$T = T \times \{0\} \xrightarrow{h} T \times \{i/k\} \subset T \times [0, 1] \xrightarrow{p} T.$$

Then $(h_1)^k = \psi$ and $h[x, t] = [h_1(x), t+1/k]$. Let $g: M \rightarrow M$ be any homeomorphism such that $g[x, t] = [g_1(x), t]$. Consider the infinite covering of $M = T \times R/\psi$ by $T \times R$. Observe that h is homotopic to g if and only if $h_1^{-1}g_1$ is homotopic (and hence isotopic) to some power of $\psi = (h_1)^k$. By hypothesis $h \simeq 1_M$. Hence some power of $(h_1)^k$ is isotopic to h_1 . It follows that the order of h_1 is relatively prime with respect to k and is equal to n , the order of ψ .

To define an $SO(2)$ action on M , we identify the circle with R/Z and denote elements as $[s]$, for $s \in R$. We define the continuous map $S^1 \times M \rightarrow M$ by $[s] \times [x, t] \mapsto [x, ns+t]$. It is easy to check that this is a well-defined effective action of $SO(2)$ without fixed points. The free Z_k action imbedded in this $SO(2)$ action is generated by the homeomorphism $\gamma: [x, t] \mapsto [x, t+1/k]$. We have $an+bk=1$ for some integers a, b . A computation shows that $\gamma^{an}[x, t] = [\psi^b(x), t+1/k] = [h_1(x), t+1/k]$. Therefore $\gamma^{an} = h$. $\square$

COROLLARY 1. *If a closed, orientable, irreducible 3-manifold M admits a proper free Z_k action ($k \geq 2$ and prime) that is Z -classified, then the orbit space M^* fibers over the circle and admits an effective $SO(2)$ action without fixed points.*

COROLLARY 2. *Suppose a closed, orientable, irreducible 3-manifold M admits a Z -classified, proper, free Z_k action ($k \geq 2$ and prime). Then there is only one such Z_k action on M (up to weak equivalence).*

PROOF. This follows from the uniqueness of the effective $SO(2)$ action on M in which such a Z_k action is imbedded [3, Theorem 4]. $\square$

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