SPAN OF LENS SPACES

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ABSTRACT. We prove that the span (i.e., maximal number of linearly independent vector fields) of $(2n+1)$ -dimensional lens spaces $L^n(p)$, where p is an odd prime and where $n+1=m \cdot 2^t$ (m odd), is equal to the span of $(2n+1)$ -dimensional sphere if $t+1 \equiv 0, 1, 2 \pmod{4}$ or if $n > 3$, $t+1 \equiv 3 \pmod{4}$, and $p \geq t+3$. This result is an improvement of a theorem given by T. Yoshida.

Let $L^n(p)$, p an odd prime, be a $(2n+1)$ -dimensional (standard) lens space. We define span of a closed connected manifold M to be the maximal number of linearly independent vector fields on M .

In this note, using an obstruction theory adopted by Denis Sjerve [1], we prove the following:

THEOREM. *Let $n+1=m \cdot 2^t$ (m odd). If $t+1 \equiv 0, 1$ or $2 \pmod{4}$, then $\text{span}(L^n(p)) = \text{span}(S^{2n+1})$. If $t+1 \equiv 3 \pmod{4}$, and $n > 3$, then $\text{span}(S^{2n+1}) - 1 \leq \text{span}(L^n(p)) \leq \text{span}(S^{2n+1})$. Moreover for $p \geq t+3$, we have $\text{span}(L^n(p)) = \text{span}(S^{2n+1})$.*

This is an improvement of a theorem of T. Yoshida [2], which asserts that

- (i) $\text{span}(S^{2n+1}) - 1 \leq \text{span}(L^n(p)) \leq \text{span}(S^{2n+1})$, if $t+1 \equiv 0 \pmod{4}$.
 - (ii) $\text{span}(L^n(p)) = \text{span}(S^{2n+1})$, if $t+1 \equiv 1, \text{ or } 2 \pmod{4}$.
 - (iii) $\text{span}(S^{2n+1}) - 2 \leq \text{span}(L^n(p)) \leq \text{span}(S^{2n+1})$, if $t+1 \equiv 3 \pmod{4}$.
- Especially, $\text{span}(L^3(3)) = 5 = \text{span}(S^7) - 2$.

PROOF OF THE THEOREM. By Yoshida's theorem, we have to prove our theorem only for the case $t+1 \equiv 0$ or $3 \pmod{4}$.

First, assume $t+1 \equiv 0 \pmod{4}$. Then $\text{span}(S^{2n+1}) = 2t+2$. Let ξ be $V_{2n+1, 2t+2}$ -bundle associated with principal $SO(2n+1)$ -bundle over $L^n(p)$. It is sufficient to prove that $\xi: L^n(p) \rightarrow BSO(2n+1)$ can be lifted to $L^n(p) \rightarrow BSO(2n+1-2t-2)$. By [1, Lemma (2.2)], $\pi_i(V_{2n+1, 2t+2})$ has no p -part for $i \leq 2n$ and is finite for $i < 4(n-t) - 1$. Since $t \geq 3$, $t+1 \equiv 0 \pmod{4}$, we have $4(n-t) - 1 > 2n$. Therefore, just as in the proof of Theorem A in [1], we find that the obstructions to the above lifting vanish.

Next we assume $t+1 \equiv 3 \pmod{4}$. In this case $\text{span}(S^{2n+1}) = 2t+3$. If $n > 3$, we have $\text{span}(L^n(p)) \geq 2t+2$, just as above. Now let ξ be

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$V_{2n+1, 2t+3}$ -bundle associated with principal $SO(2n+1)$ -bundle over $L^n(p)$. Consider the obstruction to lifting $\xi: L^n(p) \rightarrow BSO(2n+1)$ to $L^n(p) \rightarrow BSO(2n+1-2t-3)$. If $p \geq t+3$, then $\pi_i = \pi_i(V_{2n+1, 2t+3})$, $i \leq 2n$, has no p -part and is finite with the exception of $\pi_{2(n-t)-2} = \mathbb{Z}$ [1, Lemma (2.2)]. Therefore, in the diagram

$$\begin{array}{ccc}
 & & BSO(2n+1-2t-3) \\
 & & \downarrow \\
 & & E \rightarrow K(\pi_{2n}, 2n+1) \\
 & \nearrow \eta & \downarrow \\
 & & E_1 \\
 & \nearrow \eta_1 & \downarrow \\
 L^n(p) & \xrightarrow{\xi} & BSO(2n+1) \rightarrow K(\pi_{2(n-t)-2}, 2(n-t)-1),
 \end{array}$$

ξ can be lifted to η_1 , since $H^{2(n-t)-1}(L^n(p), \mathbb{Z}) = 0$. The fact that η_1 is liftable to η and η is liftable to $L^n(p) \rightarrow BSO(2n+1-2t-3)$ is easily proved as in the former case.

REFERENCES

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