HIGHER HOMOTOPY COMMUTATIVITY AND EXTENSION OF MAPS

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ABSTRACT. Let X denote the cartesian product of based spaces, $X = X_1 \times \cdots \times X_n$, and $A = X_1 \vee \cdots \vee X_n$, the subspace consisting of their one-point union. Further, let $g: A \rightarrow Y$ be a map, for Y any based space. This article develops a criterion for the extendibility of g to a map $G: X \rightarrow Y$. The criterion is in terms of higher products which live in the Pontryagin ring of ΩY , the loop space of Y .

A basic problem in topology is the extension problem: Given spaces A , X , and Y , where A is a subspace of X , and a map $g: A \rightarrow Y$, can g be extended to a map $G: X \rightarrow Y$? In the category of based spaces, an important special case is that in which X is the cartesian product $X_1 \times \cdots \times X_n$ of spaces $X_1, \cdots, X_n$ and $A = X_1 \vee \cdots \vee X_n$, the subspace consisting of their one-point union. In [6], Samelson treated this case for $n = 2$ and X_1 and X_2 being spheres, by relating the extension problem to the Pontryagin product in the homology of ΩY , the loop space of Y . In this note, Samelson's method is extended to the general situation of arbitrary $X_1, \cdots, X_n$ through the use of the higher homotopy commutativity introduced in [7]. The outline is as follows. The first section transforms the extension problem to a problem in higher topological homotopy commutativity, and the second section completes the transfer to a problem in higher algebraic homotopy commutativity. The final section illustrates the method through an application to complex Grassmann manifolds. The author wishes to acknowledge helpful suggestions from G. J. Porter.

1. We begin by recalling the basic definitions of higher homotopy commutativity.

Let K_n denote the $(n-1)$ -dimensional convex linear cell obtained by taking the convex hull of the orbit of the point $(1, \cdots, n)$ under permutation of the coordinates in R^n . If $0 < r < n$, define $d_r: K_r \times K_{n-r} \rightarrow K_n$ by

$$d_r(x_1, \cdots, x_r, y_1, \cdots, y_{n-r}) = (x_1, \cdots, x_r, y_1 + r, \cdots, y_{n-r} + r).$$

The $(n-2)$ -dimensional faces of K_n are in 1:1 correspondence with the

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set of (p, q) -shuffles, $p+q=n$. If (A_p, B_q) is such a shuffle, then $V(A_p, B_q):K_p \times K_q \rightarrow K_n$ is the homeomorphism given by $V(A_p, B_q) = \alpha^{-1} \circ d_p$ where α is the homeomorphism of R^n to itself determined by the action of (A_p, B_q) on the coordinates. For pictures of K_n , as well as further information about the K_n 's, see [3].

DEFINITION 1. Let G be an associative H -space and $Z_1, \dots, Z_n$ be pointed spaces. For each i , let $f_i: Z_i \rightarrow G$ be a map. Then a C_n -form for the f_i 's consists of a system of maps $f(i_1, \dots, i_p): K_p \times Z_{i_1} \times \dots \times Z_{i_p} \rightarrow G$, indexed by subsequences $(i_1, \dots, i_p)$ of $(1, \dots, n)$ such that

$$(1) \quad f(i) = f_i,$$

and

$$(2) \quad \begin{aligned} f(i_1, \dots, i_p)[V(A_r, B_s)(\rho, \sigma), x_1, \dots, x_p] \\ = f(i_{A(1)}, \dots, i_{A(r)})[\rho, x_{A(1)}, \dots, x_{A(r)}] \\ \cdot f(i_{B(1)}, \dots, i_{B(s)})[\sigma, x_{B(1)}, \dots, x_{B(s)}], \end{aligned}$$

where (A_r, B_s) is an (r, s) -shuffle of $(1, \dots, p)$, $(\rho, \sigma) \in K_r \times K_s$, and $x_k \in Z_{i_k}$.

REMARKS. Observe that a C_2 -form for two maps f_1 and f_2 is just a homotopy between the product maps $f_1 \cdot f_2$ and $f_2 \cdot f_1$. For further intuition into C_n -commutativity, the reader is referred to p. 193 of [7]. In that paper is defined the notion of C_n -space: An associative H -space is a C_n -space provided that there is a C_n -form for the maps $f_1, \dots, f_n$ where each $f_i = 1: G \rightarrow G$. Clearly, if G is a C_n -space, then any maps $f_i: Z_i \rightarrow G$, $i = 1, \dots, n$, possess a C_n -form.

Now consider maps $\tilde{f}_i: SZ_i \rightarrow Y$, $i = 1, \dots, n$, where SZ_i denotes the suspension of the space Z_i . Then the map $\tilde{f}: SZ_1 \vee \dots \vee SZ_n \rightarrow Y$ is defined by $\tilde{f}|SZ_i = \tilde{f}_i$. Suppose that f extends to $F: SZ_1 \times \dots \times SZ_n \rightarrow Y$. Then for each subsequence $(i_1, \dots, i_p)$ of $(1, \dots, n)$ a map $F(i_1, \dots, i_p): I^p \times Z_{i_1} \times \dots \times Z_{i_p} \rightarrow Y$ is defined by the composition

$$\begin{aligned} I^p \times Z_{i_1} \times \dots \times Z_{i_p} &\xrightarrow{q} I \times Z_{i_1} \times \dots \times I \times Z_{i_p} \\ &\xrightarrow{r} SZ_{i_1} \times \dots \times SZ_{i_p} \xrightarrow{s} Y, \end{aligned}$$

where q interchanges the factors, r is the identification map, and $s = F|SZ_{i_1} \times \dots \times SZ_{i_p}$. Now let $\hat{f}(i_1, \dots, i_p): [0, p] \times K_p \times Z_{i_1} \times \dots \times Z_{i_p} \rightarrow Y$ be defined as follows.

First, let maps $\tau_p: [0, p] \times K_p \rightarrow I^p$ be defined inductively as follows. Let $\tau_1: [0, 1] \times K_1 \rightarrow I$ be the identity map $\tau_1(t, *) = t$. Suppose τ_k is

defined for $k < p$. Define τ_p from $\text{Bd}([0, p] \times K_p) \rightarrow \text{Bd}(I^p)$ by $\tau_p(0, \kappa) = (0, \dots, 0)$, $\tau_p(1, \kappa) = (1, \dots, 1)$, and

$$\begin{aligned}\tau_p(t, V(A_r, B_s)(\rho, \sigma)) &= [A_r, B_s](\tau_r(t, \rho), 0, \dots, 0), & \text{for } 0 \leq t \leq r, \\ &= [A_r, B_s](1, \dots, 1, \tau_s(t - r, \sigma)), & \text{for } r \leq t \leq p,\end{aligned}$$

where $\kappa \in K_p$, $(\rho, \sigma) \in K_r \times K_s$, $t \in [0, p]$, and $[A_r, B_s]$ denotes the shuffle acting on I^p by permuting the coordinates. The definition of τ_p is completed by regarding $[0, p] \times K_p$ and I^p as the cones on their respective boundaries and extending τ_p by linearity. Now let $\hat{f}(i_1, \dots, i_p)$ be defined to be the composition $F(i_1, \dots, i_p) \circ (\tau_p \times \text{id})$. Finally, define $f(i_1, \dots, i_p): K_p \times Z_{i_1} \times \dots \times Z_{i_p} \rightarrow \Omega Y$ by the formula

$$f(i_1, \dots, i_p)(\tau, z_{i_1}, \dots, z_{i_p})[t] = \hat{f}(i_1, \dots, i_p)(t, \tau, z_{i_1}, \dots, z_{i_p}).$$

(Here ΩY denotes the associative model of the loop space on Y .) It is immediate from the construction that the $f(i_1, \dots, i_p)$'s yield a C_n -form for $f_1, \dots, f_n$. This discussion is summarized in the following theorem.

THEOREM 2. *If a map $\tilde{f}: SZ_1 \vee \dots \vee SZ_n \rightarrow Y$ extends to $F: SZ_1 \times \dots \times SZ_n \rightarrow Y$, then there exists a C_n -form $f(i_1, \dots, i_p)$ for the maps $f_1, \dots, f_n$ obtained as above.*

To return to the general case, let $X_1, \dots, X_n$ be spaces, $g_i: X_i \rightarrow Y$, $i = 1, \dots, n$, be maps, and $g: X_1 \vee \dots \vee X_n \rightarrow Y$ be defined by $g|X_i = g_i$. Suppose that g extends to $G: X_1 \times \dots \times X_n \rightarrow Y$. Then for any suspensions $SZ_1, \dots, SZ_n$ and maps $\phi_i: SZ_i \rightarrow X_i$, the composition $\tilde{f}$ given by

$$SZ_1 \vee \dots \vee SZ_n \xrightarrow{\phi_1 \vee \dots \vee \phi_n} X_1 \vee \dots \vee X_n \xrightarrow{g} Y$$

extends to $F: SZ_1 \times \dots \times SZ_n \rightarrow Y$. Thus the following corollary holds.

COROLLARY 3. *Let $g: X_1 \vee \dots \vee X_n \rightarrow Y$ be a map. If g extends to a map $G: X_1 \times \dots \times X_n \rightarrow Y$, then for any choice of spaces and maps $\phi_i: SZ_i \rightarrow X_i$, there is a C_n -form for the maps $f_i: Z_i \rightarrow \Omega Y$ obtained as above.*

REMARK A. Although the corollary may seem rather unwieldy, in practice there is always a good candidate for the ϕ_i , namely the evaluation map $e_i: S\Omega X_i \rightarrow X_i$. For this choice, the maps f_i are just the looped maps $\Omega g_i: \Omega X_i \rightarrow \Omega Y$.

REMARK B. In [5], the higher Whitehead product set $[f_1, \dots, f_n]$ was defined, which measured the obstructions to extending $\bar{f}$ to F as in Theorem 2. Analogously, a higher Samelson product set $\langle f_1, \dots, f_n \rangle$ may be constructed which measures the obstructions to the existence of a C_n -form for $f_1, \dots, f_n$. In a future paper, the author will construct $\langle f_1, \dots, f_n \rangle$ and examine its relationship to $[f_1, \dots, f_n]$ and to the higher homology commutator products discussed below.

2. In order to transfer our situation to homology groups wherein calculations can be carried out, it is necessary to introduce the homology version of higher homotopy commutativity. Let G be an associative H -space.

DEFINITION 4. Let $a_i \in C_{n_i}(G)$, $i=1, \dots, n$, be cycles. Then a *determining system for a higher commutator product of the a_i 's* is a system of elements $a(i_1, \dots, i_p)$ of $C_*(G)$, indexed by proper subsequences of $(1, \dots, n)$, such that

$$(1) \quad a(i) = a_i,$$

and

$$(2) \quad \begin{aligned} & \partial a(i_1, \dots, i_p) \\ &= \sum_{(A_r, B_s)} (-1)^{\epsilon(A_r, B_s)} a(i_{A(1)}, \dots, i_{A(r)}) \quad a(i_{B(1)}, \dots, i_{B(s)}), \end{aligned}$$

where the summation is taken over all shuffles of $(1, \dots, p)$ and

$$\begin{aligned} \epsilon(A_r, B_s) &= \text{sgn}(A_r, B_s) + (s-1)(1 + \deg a(i_{A(1)}, \dots, i_{A(r)})) \\ &+ \sum_{B(j) < A(k)} \deg a(i_{B(j)}) \deg a(i_{A(k)}). \end{aligned}$$

The higher commutator product determined by this system is the homology class of the cycle

$$\begin{aligned} & \bar{a}(1, \dots, n) \\ &= \sum_{(A_r, B_s)} (-1)^{\epsilon(A_r, B_s)} a(A(1), \dots, A(r)) \quad a(B(1), \dots, B(s)). \end{aligned}$$

By checking the various signs involved, it is easy to verify that $\bar{a}(1, \dots, n)$ is, in fact, a cycle. The sign is chosen to make Proposition 5 below hold. Observe that if $a_1, \dots, a_n$ are all odd-dimensional, then all the signs involved are $+1$.

The set of all elements obtained in this way from $a_1, \dots, a_n$ is called the *higher commutator product set* of $a_1, \dots, a_n$ and denoted by

$\langle a_1, \dots, a_n \rangle$. It is a subset of $H_N(G)$, where $N = \sum_{i=1}^n \deg a_i + n - 2$, is natural with respect to homomorphisms of monoids, and depends only upon the homology classes of $a_1, \dots, a_n$.

Now let $Z_1, \dots, Z_n$ be spaces and $f_i: Z_i \rightarrow G$, $i = 1, \dots, n$, be maps. For each i , let α_i be an element of $H_*(Z_i)$ and $a_i = f_{i*}(\alpha_i)$. The next proposition follows from Definitions 1 and 4.

PROPOSITION 5. *In the above situation, if there exists a C_n -form for the maps $f_1, \dots, f_n$, then 0 is an element of $\langle a_1, \dots, a_n \rangle$.*

PROOF. Define $a(i_1, \dots, i_p) = f(i_1, \dots, i_p) \# [\kappa_p \otimes \alpha_{i_1} \otimes \dots \otimes \alpha_{i_p}]$. Here κ_p is the fundamental cell defined in [3]. The signs needed come out of the formula for $\partial(\kappa_p)$ given in [3] and from the usual signs involved in permuting factors in tensor products.

Note. A consequence of Proposition 5 is that if G is a C_n -space, then all $\langle a_1, \dots, a_k \rangle$, $k \leq n$, contain 0. The commutator product version of Lemma 20 of [2] holds in this case to guarantee that if $a_1, \dots, a_{n+1}$ are all odd-dimensional, then $\langle a_1, \dots, a_{n+1} \rangle$ consists of a single element.

The following criterion now results from combining Corollary 3 with Proposition 5.

THEOREM 6. *Suppose the following conditions hold:*

- (1) $g: X_1 \vee \dots \vee X_n \rightarrow Y$ is a map and $g_i = g|X_i$, $i = 1, \dots, n$;
- (2) $Z_1, \dots, Z_n$ are spaces and $\phi_i: SZ_i \rightarrow X_i$, $i = 1, \dots, n$, are maps;
- (3) for $i = 1, \dots, n$, $f_i: Z_i \rightarrow \Omega Y$ is the adjoint to $\tilde{f}_i = g_i \circ \phi_i: SZ_i \rightarrow Y$; and
- (4) for $i = 1, \dots, n$, $\alpha_i \in H_*(Z_i)$ and $a_i = f_{i*}(\alpha_i)$.

Then if g extends to $G: X_1 \times \dots \times X_n \rightarrow Y$, 0 is an element of $\langle a_1, \dots, a_n \rangle$.

REMARK 7. From an examination of the proof of Proposition 5, we see that if g extends to the cartesian product, then 0 is actually an element of the subset S of $\langle a_1, \dots, a_n \rangle$ consisting of those higher product elements whose defining systems are made up of elements which possibly might be obtained as images of maps $f(i_1, \dots, i_p) \#$.

The next section gives an illustration of how this criterion may be applied to a particular extension problem.

3. Denote by $G_{n,k}$ the complex Grassmann manifold of k -planes in complex $(k+n)$ -space. Given an integer k , consider the inclusion $i: G_{k,k} \subset G_{3k-1, 3k-1}$. Let a map $g: G_{k,k} \vee G_{k,k} \vee G_{k,k} \rightarrow G_{3k-1, 3k-1}$ be defined to be i on each component. Let $j: G_{3k-1, 3k-1} \subset G_{3k, 3k}$ be the

inclusion. Then it is well known that jg can be extended to map $G_{k,k} \times G_{k,k} \times G_{k,k}$ into $G_{3k,3k}$ (this is the way the multiplication is defined to make BU , the classifying space for the infinite unitary group, into an H -space). Here it will be shown that this is the best possible dimension, i.e. that g itself does not extend.

To recreate the situation of Theorem 6, let $Y = G_{3k-1, 3k-1}$, $X_i = G_{k,k}$, $Z_i = S^{2k-1}$, and $\phi_i: S^{2k} \rightarrow X_i$ represent a generator of $\pi_{2k}(G_{k,k})$, for $i = 1, 2, 3$. Thus in this case, each f_i represents a generator of $\pi_{2k-1}(\Omega Y)$. We choose each $\alpha_i \in H_*(S^{2k-1})$ to be the generator in dimension $2k-1$, thus $a_i = f_{i*}(\alpha_i)$ is known to be $(k-1)!$ times the generator of $H_{2k-1}(\Omega Y)$. In order to compute the higher product set $\langle a_1, a_2, a_3 \rangle$, we need a good chain complex at hand to give the homology of ΩY . The cobar construction fills the bill.

Recall that the ring structure of $H^*(G_{l,m})$ is obtained by taking the graded ring of formal power series Γ on generators $\sigma_1, \dots, \sigma_m$, $\dim(\sigma_i) = 2i$, and adding the relations $\bar{\sigma}_i = 0$, $i > l$, defined as follows. Set $\sigma = 1 + \sigma_1 + \dots + \sigma_m$, let $\bar{\sigma}$ be the formal inverse of σ in Γ , and then let $\bar{\sigma}_i$ be the component of $\bar{\sigma}$ of degree $2i$.

The cobar construction for $H_*(\Omega Y)$ is begun by forming the tensor algebra on generators $\{u\}$ in dimensions one lower than the generators $\{t\}$ of $H_*(Y)$. The differential operator, d , is defined as follows. Let Δ be the comultiplication in $H_*(Y)$. If u corresponds to t , then define $d(u) = \sum u_i \otimes u'_i$, where $\Delta(t) = \sum t_i \otimes t'_i$ and the t_i and t'_i correspond to u_i and u'_i respectively. The complex thus obtained is chain equivalent to the singular chain complex of ΩY , cf [1].

In $H_*(Y)$, let $t(i_1, \dots, i_p)$ denote the element dual to $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_p}$, $i_1 \leq i_2 \leq \dots \leq i_p$. In the cobar construction, let $u(i_1, \dots, i_p)$ correspond to $t(i_1, \dots, i_p)$. A straightforward computation yields that at the chain level, there is a nonzero integer b such that $\langle u(i_1), \dots, u(i_p) \rangle = b \cdot d(u(i_1, \dots, i_p))$, if $\deg(\langle u(i_1), \dots, u(i_p) \rangle) < 6k-2$. Now each element a_i is represented at the chain level by $(k-1)!$ times $u(k)$. The determining system induced by the multiplication map $G_{k,k} \times G_{k,k} \rightarrow G_{2k,2k}$ yields the element $v = 6(k-1)!(u(k,k)u(k,k) + u(k)u(k,k))$ as an element of $\langle a_1, a_2, a_3 \rangle$. The relation $\bar{\sigma}_{3k} = 0$ implies that this element is not a boundary. According to Remark 6, we need only consider elements of $\langle a_1, a_2, a_3 \rangle$ which arise from maps $K_2 \times S^{2k-1} \times S^{2k-1} \rightarrow \Omega Y$. But any such element differs from v by a multiple of the element $(k-1)!(2k-1) \cdot (u(k)u(2k) + u(2k)u(k))$, and again, the relation $\bar{\sigma}_{3k} = 0$ implies that no such element is a boundary. Hence g does not extend to the cartesian product.

REFERENCES

1. J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 409–412. MR **18**, 59.
2. D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449. MR **34** #2010.
3. R. J. Milgram, *Iterated loop spaces*, Ann. of Math. (2) **84** (1966), 386–403. MR **34** #6767.
4. G. J. Porter, *Higher order Whitehead products and Postnikov systems*, Illinois J. Math. **11** (1967), 414–416.
5. ———, *Higher-order Whitehead products*, Topology **3** (1965), 123–135. MR **30** #4261.
6. H. Samelson, *A connection between the Whitehead and the Pontryagin product*, Amer. J. Math. **75** (1953), 744–752. MR **15**, 731.
7. F. D. Williams, *Higher homotopy-commutativity*, Trans. Amer. Math. Soc. **139** (1969), 190–206. MR **39** #2163.

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