UNIQUENESS OF GENERATORS OF PRINCIPAL IDEALS IN RINGS OF CONTINUOUS FUNCTIONS

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ABSTRACT. Let aR denote the principal right ideal generated in a ring R by an element a . Kaplansky has raised the question: If $aR = bR$, are a and b necessarily right associates? In this note we show that for rings of continuous functions the answer is affirmative if and only if the underlying topological space is zero-dimensional. This gives an algebraic characterization of the topological concept "zero-dimensional". By extending the notion of uniqueness of generators of principal ideals we are able to give an algebraic characterization of the concept " n -dimensional".

In the sequel, R is assumed to be commutative.

DEFINITION. A set of principal ideals $\{a_i R\}$, $i = 1, \dots, n$, is *uniquely generated* if whenever $a_i R = b_i R$, $i = 1, \dots, n$, there exist elements u_i of R such that $a_i = b_i u_i$, $i = 1, \dots, n$, and $u_1 R + \dots + u_n R = R$. The *dimension* of R —denoted by $\dim R$ —is the least integer n such that every set of $n+1$ principal ideals is uniquely generated.

In the following X will denote a completely regular topological space, $C(X)$ the ring of real-valued continuous functions defined on X , and $C^*(X)$ the subring of $C(X)$ consisting of the bounded functions in $C(X)$. For $f \in C(X)$, the zero set $Z(f)$ of f , is defined by $Z(f) = \{x \in X : f(x) = 0\}$. Clearly $f_1 C(X) + \dots + f_n C(X) = C(X)$ if and only if $Z(f_1) \cap \dots \cap Z(f_n) = \emptyset$. For further information on zero sets the reader is referred to [2]. An important fact is that disjoint zero sets are completely separated.

We use the modification of covering dimension involving basic covers [2, p. 243], and the equivalent definitions given in [1]. The unit cube in E^{n+1} is denoted by

$$I^{n+1}:I^{n+1} = \{x \in E^{n+1} : -1 \leq x_i \leq 1, i = 1, \dots, n+1\}.$$

We also write

$$I_+^{n+1} = \{x \in E^{n+1} : 0 \leq x_i \leq 1, i = 1, \dots, n+1\}$$

and

$$S_+^n = \{x \in I_+^{n+1} : x_i = 0 \text{ or } 1 \text{ for some } i\}.$$

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THEOREM. *The following conditions are equivalent.*

- (i) $\dim X = n$.
- (ii) $\dim C(X) = n$.
- (iii) $\dim C^*(X) = n$.

PROOF. $\dim C(X) \leq \dim X$. Let $\dim X = n$ and suppose $f_i C(X) = g_i C(X)$, $i = 1, \dots, n+1$. There exist functions s_i, t_i in $C(X)$ such that $f_i = g_i s_i$, $g_i = f_i t_i$, $i = 1, \dots, n+1$. Then $g_i = g_i s_i t_i$, so that $g_i(1 - s_i t_i) = 0$. Since $Z_i = Z(s_i)$ and $Z'_i = Z(s_i + 1 - s_i t_i)$ are disjoint zero sets, there exist m_i in $C(X)$ such that m_i is 1 on Z_i , 0 on Z'_i and $0 \leq m_i \leq 1$. Let m be the mapping of X into I_+^{n+1} defined by $m(x) = (m_1(x), \dots, m_{n+1}(x))$. Then m maps all the points in Z_i and Z'_i into S_+^n . Since $\dim X \leq n$, there exists a mapping $h = (h_1, \dots, h_{n+1})$ of X into S_+^n such that $h(x) = m(x)$ whenever $m(x) \in S_+^n$ [1, Definition 3]. Let $k_i = s_i + h_i(1 - s_i t_i)$, $i = 1, \dots, n+1$. Then $Z(k_1) \cap \dots \cap Z(k_{n+1}) = \emptyset$. To see this, note that for any $x \in X$ we have $h_i(x) = 0$ or $h_i(x) = 1$ for some i . If $h_i(x) = 0$ then $s_i(x) \neq 0$ so that $k_i(x) \neq 0$. If $h_i(x) = 1$, then $s_i(x) + 1 - s_i(x)t_i(x) \neq 0$ and again $k_i(x) \neq 0$. Hence $k_i C(X) + \dots + k_{n+1} C(X) = C(X)$. Finally, $g_i k_i = g_i s_i + h_i g_i(1 - s_i t_i) = g_i s_i = f_i$, $i = 1, \dots, n+1$.

$\dim X \leq \dim C(X)$. Let $\dim C(X) = n$ and let Z_i, Z'_i , $i = 1, \dots, n+1$, be disjoint pairs of zero sets of X . We construct functions f_i, k_i such that $f_i = k_i |f_i|$ and $f_i = 1$ on Z_i , $f_i = -1$ on Z'_i , $i = 1, \dots, n+1$. There exist p_i such that $p_i = 1$ on Z_i and $p_i = -1$ on Z'_i . There exist k_i, s_i, t_i such that $k_i = 1$ when $p_i \geq \frac{1}{2}$, $k_i = -1$ when $p_i \leq -\frac{1}{2}$, $-1 \leq k_i \leq 1$:

$$\begin{aligned} t_i &= 1 & \text{on } Z_i, t_i &= 0 & \text{when } p_i \leq \frac{1}{2}, 0 \leq t_i \leq 1, \\ s_i &= -1 & \text{on } Z'_i, s_i &= 0 & \text{when } p_i \geq -\frac{1}{2}, -1 \leq s_i \leq 0. \end{aligned}$$

Let $f_i = s_i + t_i$. Then $f_i = k_i |f_i|$, $|f_i| = k_i f_i$, so that $f_i C(X) = |f_i| C(X)$, $i = 1, \dots, n+1$. Since $\dim C(X) \leq n$, there exist $h_1, \dots, h_{n+1}$ such that $f_i = h_i |f_i|$, $i = 1, \dots, n+1$, and $h_1 C(X) + \dots + h_n C(X) = C(X)$. Clearly Z_i and Z'_i are separated in $X - Z(h_i)$ and $Z(h_1) \cap \dots \cap Z(h_{n+1}) = \emptyset$. Hence $\dim X \leq n$ [1, Definition 4].

$\dim X = \dim C^*(X)$. This is proved using the same methods as above, with $C(X)$ replaced by $C^*(X)$.

A simple consequence of this theorem is the following well-known result.

COROLLARY. $\dim X = \dim \beta X = \dim vX$, where βX is the Stone-Ćech compactification of X , and vX is the Hewitt real-compactification of X .

PROOF. Since $C^*(X)$ is isomorphic to $C(\beta X)$, we have $\dim X$

$=n \Leftrightarrow \dim C^*(X) = n \Leftrightarrow \dim C(\beta X) = n \Leftrightarrow \dim \beta X = n$. Since $C(X)$ is isomorphic to $C(vX)$, we have $\dim X = n \Leftrightarrow \dim C(X) = n \Leftrightarrow \dim C(vX) = n \Leftrightarrow \dim vX = n$.

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