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C^∞-FUNCTIONS NEED NOT BE BIMEASURABLE[1]

R. B. DARST

ABSTRACT. A real valued C^∞-function f is constructed on the interval $I=[0, 1]$ such that some Borel subsets of I are mapped by f onto non-Borel sets.

A Borel function g whose domain is a Borel subset V of a separable complete metric space $\mathfrak{M}_1$ and whose range is contained in a separable complete metric space $\mathfrak{M}_2$ is said to be bimeasurable if g maps Borel subsets of V onto Borel subsets of $\mathfrak{M}_2$.

We shall construct a real valued infinitely differentiable function f defined on the interval I which is not bimeasurable. All the derivatives of f at both zero and one are zero, so f can be extended to a nonbimeasurable C^∞-function on R^1 by setting $f(x)=f(0)$ when $x<0$ and $f(x)=f(1)$ when $x>1$. Then f can be extended to a nonbimeasurable C^∞ function on R^2 by setting $f(x, y)=f(x, 0)=f(x)$. These functions illustrate another contrast between locally analytic functions and functions which are merely infinitely differentiable because locally analytic functions are bimeasurable. For the sake of completeness, we shall substantiate this latter assertion before beginning our construction. To this end, suppose that $\mathfrak{M}_1$ is either R^1 or R^2, V is an open subset of $\mathfrak{M}_1$, and $\mathfrak{M}_2=R^2$. Suppose that g is a Borel function mapping V into R^2. Moreover, suppose that for each element x of V, there exists a positive number ϵ_x and a sequence $\{a_{x,j}\}_{j=0}^\infty$ of complex numbers such that $g(x)=a_{x,0}$ and if $0<|y-x|<\epsilon_x$, then $g(y)=\sum_{j=0}^\infty a_{x,j}(y-x)^j$. An open subset V of a metric space $\mathfrak{M}_1$ is a countable union $\bigcup_{j=1}^\infty V_j$ of closed subsets V_j of $\mathfrak{M}_1$. If S is an uncountable subset of V, then $S_j=S\cap V_j$ is uncountable for some positive integer j. Since $\mathfrak{M}_1$ is separable and complete, S_j has a limit point in V_j. For each point p in R^2, denote $g^{-1}(p)$ by S_p. If S_p is uncountable, let x_p be a limit point of S_p in V. Then g takes the constant value p on the open ball $\{y; |y-x_p|<\epsilon_{x_p}\}$. There are only countably many pairwise disjoint open sets in a separable metric space, so $U(g)=\{p; g^{-1}(p) \text{ is uncountable}\}$ is a countable set. Since countable

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sets are Borel sets and g is a Borel function, $g^{-1}(U(g))$ is a Borel subset of V and the restriction of g to $g^{-1}(U(g))$ is bimeasurable. The restriction g_1 of g to $V-g^{-1}(U(g))$ has the property that the inverse image $g_1^{-1}(p)$ is a countable set for each $p\in R^2$, so a theorem of Lusin [3, pp. 237–252] implies that g_1 maps Borel sets onto Borel sets. Thus g is bimeasurable.

A brief history of bimeasurable functions and their relations to measure theory can be found in [1].

Turning now to a construction of our example, we begin by reminding the reader that in [4], Roger Purves showed that a Borel function ϕ on I is bimeasurable if, and only if,

$$\operatorname{card}\{y;\ \operatorname{card}[\phi^{-1}(y)] > \aleph_0\} \leqq \aleph_0.$$

So, in order to establish our example, it suffices to construct f so that there is a Cantor set F in the range, $f(I)$, of f such that if $y\in F$, then $f^{-1}(y)$ contains a Cantor set. Our strategy is to construct f so that it oscillates and repeats its parts in a systematic fashion, to be explained in the sequel. In order to help explain why f has its alleged properties, we shall first look at a continuous function h of bounded variation which is not bimeasurable and has a direct and elementary construction. Arthur H. Stone very kindly showed the function h to the author in connection with another problem [2]. Define h on the usual Cantor ternary set C by

$$h\left(\sum_{n=1}^{\infty} a_n\cdot 3^{-n}\right) = \sum_{n=1}^{\infty} a_{2n}\cdot 9^{-n}$$

(where each of $a_1, a_2, \cdots$ is 0 or 2) and extend h to I by making it linear on each complementary interval. An elementary calculation then shows

$$|h(x) - h(y)| \leqq 3|x-y| \qquad \text{for all } x, y \in I,$$

so that h is continuous and of bounded variation; and clearly card $[h^{-1}(y)]=c$ for each of the c numbers in $h(C)$. However, for our purposes it is helpful to look at h geometrically rather than algebraically. To this end, let I_{ij}, $j\leqq 2^i$, denote the ith stage intervals in the canonical representation of C as an intersection of finite unions of intervals, and let O_{ij}, $j\leqq 2^{i-1}$, denote the corresponding segments which are removed at the ith stage. Notice that h has the following properties:

(i) $h(0)=0$;

(ii) h repeats itself on I_{11} and I_{12} (i.e., $h(x-2/3)=h(x)$, $2/3\leqq x\leqq 1$),

$h(I_{21})$ lies below $h(I_{22})$, h repeats itself on each of the pairs I_{31}, I_{32} and I_{33}, I_{34}, $\cdots$;

(iii) the set

$$H = [h(I_{21}) \cup h(I_{22})] \cap [h(I_{41}) \cup h(I_{42}) \cup h(I_{45}) \cup h(I_{46})] \cap \cdots$$

is a Cantor set, and if $y \in H$, then by "tracing back" one sees that $h^{-1}(y)$ contains a Cantor set;

(iv) given the definition of h on the segments O_{ij} there is at most one continuous extension to I which satisfies (i); moreover, h is sufficiently smooth on the segments O_{ij} that it can be so extended;

(v) h is decreasing on the sets $O_{(2i-1)j}$ and increasing on the sets $O_{(2i)j}$.

We will define f on I so that (i)–(v) are satisfied with h and H replaced by f and F. Then we will show that f is C^∞ on I.

For $x > 0$, denote $e^{-1/x}$ by $u(x)$, and let

$$V(a, b, t) = \int_a^t u(x-a)u(b-x)dx, \qquad 0 \leqq a \leqq t \leqq b \leqq 1.$$

Notice that $u(cx) = [u(x)]^{1/c}$,

$$V(a, a+b, a+x) = V(0, b, x),$$

and

$$V(0, b, b) = 2V(0, b, b/2).$$

Let $y(b)$ denote $V(0, b, b/2)$. Then, for $6b \leqq 1$,

$$\begin{aligned} y(2b) &= \int_0^b u(t)u(2b-t)dt \\ &= (1/3)\int_0^{3b} u(x/3)u(2b-(x/3))dx \\ &= (1/3)\int_0^{3b} [u(x)]^3[u(6b-x)]^3dx \\ &< (1/3)[u(1/2)]^2\int_0^{3b} u(x)u(6b-x)dx \\ &= 1/(3e^4)y(6b) \\ &< 1/4y(6b) \end{aligned}$$

which implies

$$(*) \qquad \sum_{k=1}^{\infty} 2^k y(3^{-(n+k)}) < 1/2y(3^{-n}), \qquad n = 1, 2, \cdots.$$

For $n \geqq 1$, let $\alpha_n = 2y(3^{-n}) = V(0, 3^{-n}, 3^{-n})$. We shall determine a sequence $\{\beta_n\}$ of numbers in $(0, 1]$ such that the increase of f on each of the nth stage segments, O_{nj}, is $(-1)^n\gamma_n$, where $\gamma_n = \alpha_n\beta_n$. To maintain the behavior cited in (i) and (ii), the following sequence of equations must be satisfied:

$$\gamma_{2i-1} = \gamma_{2i} + 2\gamma_{2i+1}, \qquad i = 1, 2, \cdots.$$

These relations imply

$$\gamma_{2i-1} = \sum_{k=0}^{\infty} 2^k \gamma_{2(i+k)}, \qquad i = 1, 2, \cdots.$$

Let $\beta_{2n} = 1$, $n \geqq 1$, to obtain $\gamma_{2n} = \alpha_{2n}$. Then (*) implies that

$$\beta_{2i-1} = \left(\sum_{k=0}^{\infty} 2^k \alpha_{2(i+k)} \right) / \alpha_{2i-1} < 1.$$

The definition of f begins as follows. Let

$$\begin{aligned}
f(0) &= 0, \\
f(x) &= \beta_1(\alpha_1 - V(1/3, 2/3, x)), \qquad 1/3 \leqq x \leqq 2/3, \\
f(x) &= f(x - 2/3), \qquad 2/3 < x \leqq 1, \\
f(x) &= \beta_3(\alpha_3 - V(1/27, 2/27, x)), \qquad 1/27 \leqq x \leqq 2/27, \\
f(x) &= f(x - 2/27), \qquad 2/27 < x \leqq 3/27, \\
f(x) &= \gamma_3 + V(1/9, 2/9, x), \qquad 1/9 < x \leqq 2/9, \\
f(x) &= \gamma_3 + \alpha_2 + \beta_3(\alpha_3 - V(7/27, 8/27, x)), \qquad 7/27 < x \leqq 8/27, \\
f(x) &= f(x - 2/27), \qquad 8/27 < x \leqq 1/3, \cdots.
\end{aligned}$$

To show that f is C^∞ on I, it suffices to show that $f^{(k)}(x) = 0$, whenever k is a positive integer and $x \in C$. However, for each pair k, n of nonnegative integers, the binomial theorem, the product rule for derivatives, and bounds on the first $k+n$ derivatives of u on I imply that there exists a constant $M_{k,n}$ such that for $0 \leqq a < x < b \leqq 1$,

$$(**) \qquad \left| \frac{d^k}{dx^k} V(a, b, x) \right| \leqq M_{k,n}(x - a)^n (b - x)^n.$$

Recall that if p_i is a sequence of positive numbers, then $\sum p_i^2 < (\sum p_i)^2$. If $k = 0$, $n = 2$, and $x \in C$, then $|f(y) - f(x)| \leqq M_{0,2}|y - x|^2$

which implies that $f^{(1)}(x)=0$, $x\in C$. Suppose that $f^{(k)}(x)=0$, $x\in C$ and let $t\in C$, then $|f^{(k)}(y)-f^{(k)}(t)|\leqq M_{k,2}|y-x|^2$ which implies that $f^{(k+1)}_{(t)}=0$, $t\in C$.

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Purdue University, Lafayette, Indiana 47907