

## ON A GENERALISATION OF HERMITE POLYNOMIALS

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**ABSTRACT.** In this paper, the author introduces a generalisation of the Hermite polynomials. Hypergeometric representations, a new generating relation and  $n$ th order differential formulae for the generalised polynomials have also been derived therein.

**1. Introduction.** In the present paper, an effort has been made to generalise Hermite polynomials. For brevity, the following notations have throughout been adopted:

$$(i) \quad \Delta(b; a) = \frac{a}{b}, \frac{a+1}{b}, \dots, \frac{a+b-1}{b}.$$

$$(ii) \quad \Delta_k[b; a] = \left(\frac{a}{b}\right)_k \cdot \left(\frac{a+1}{b}\right)_k \cdots \left(\frac{a+b-1}{b}\right)_k.$$

$$(iii) \quad \Delta[a(1); b] = \frac{b}{a}, \frac{b+1}{a}, \dots, \frac{b+a-2}{a}$$

i.e. the last one term of the sequence  $\Delta(a; b)$  is deleted.

$$(iv) \quad \Delta_k[a(1); b] = \left(\frac{b}{a}\right)_k \cdot \left(\frac{b+1}{a}\right)_k \cdots \left(\frac{b+a-2}{a}\right)_k.$$

**2. Definition.** We defined the generalised polynomial set  $H_{n,m,\nu}(x)$ , by means of the generating relation:

$$(2.1) \quad \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x)t^n}{n!} = e^{xt-t^m},$$

where  $n$  is a nonnegative integer and  $m$  is a positive integer.

**3. Hypergeometric form.** Relation (2.1) can be put into the form

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x)t^n}{n!} &= \sum_{n,m=0}^{\infty} \frac{(-1)^k (\nu x)^n t^{n+m k}}{n! k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (\nu x)^{n-m k} t^n}{k! (n-m k)!}. \end{aligned}$$

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On comparing coefficients of  $t^n$  from both the sides, we get:

$$(3.1) \quad H_{n,m,\nu}(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k n! (\nu x)^{n-mk}}{k!(n-mk)!}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{(m+1)k} (-n)_{mk} (\nu x)^{n-mk}}{k!}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{(m+1)k} m^{mk} \Delta_k [(m; -n)] (\nu x)^{n-mk}}{k!}$$

$$(3.2) \quad \therefore H_{n,m,\nu}(x) = (\nu x)^n {}_mF_0 [\Delta(m; -n); -; -(-m/\nu x)^m].$$

*Particular cases.* (i) when  $m=1$ , we have

$$(3.3) \quad H_{n,1,\nu}(x) = (\nu x - 1)^n.$$

(ii) When  $m=\nu=2$ , we get the well-known result:

$$H_n(x) = (2x)^n {}_2F_0 \left[ \Delta(2; -n); -; -\frac{1}{x^2} \right].$$

4. Another hypergeometric form for  $H_{n,m,\nu}(x)$ . Taking  $n=m+m_1+m_2$  in (3.1), where  $m_1$  and  $m_2$  are positive integers and  $m_2 < m$ ,  $m$  being a positive integer, we get:

$$H_{mm_1+m_2,m,\nu}(x) = \sum_{k=0}^{m_1} \frac{(-1)^k (mm_1 + m_2)! (\nu x)^{mm_1+m_2-mk}}{k! (mm_1 + m_2 - mk)!}$$

$$= \sum_{k=0}^{m_1} \frac{(-1)^{m_1-k} (mm_1 + m_2)! (\nu x)^{m_2+mk}}{(m_1 - k)! (m_2 + mk)!}$$

$$= \frac{(-1)^{m_1} (mm_1 + m_2)! (\nu x)^{m_2}}{m_1! m_2!} \sum_{k=0}^{m_1} \frac{(-m_1)_k (\nu x)^{mk}}{(1+m_2)_{mk}}$$

$$= \frac{(-1)^{m_1} (mm_1 + m_2)! (\nu x)^{m_2}}{m_1! m_2!} \sum_{k=0}^{m_1} \frac{(-m_1)_k (1)_k (\nu x)^{mk}}{k! m^{mk} \Delta_k(m; 1+m_2)}$$

$$\therefore H_{mm_1+m_2,m,\nu}(x) = \frac{(-1)^{m_1} (1+m_2)_{mm_1} (\nu x)^{m_2}}{m_1!} \times {}_2F_m \left[ -m_1, 1; \Delta(m; 1+m_2); \left(\frac{\nu x}{m}\right)^m \right].$$

*Particular cases.*

*Case (i).* When  $m=\nu=2$ , we get

$$(4.2) \quad H_{2m_1+m_2}(x) = \frac{(-1)^{m_1}(1+m_2)_{2m_1}(2x)^{m_2}}{m_1!} \\ \times {}_2F_2[-m_1, 1; \Delta(2; 1+m_2); x^2].$$

*Case (ii).* Writing  $r$  for  $m_1$  and taking  $m_2=0$  in Case (i), we get:

$$(4.3) \quad H_{2r}(x) = (-1)^r 2^{2r} r! L_r^{(-1/2)}(x^2),$$

where  $L_r^{(\alpha)}(x)$  is the Laguerre polynomial.

*Case (iii).* Writing  $r$  for  $m_1$  and taking  $m_2=1$  in Case (i), we get:

$$(4.4) \quad H_{2r+1}(x) = (-1)^r 2^{2r+1} r! x L_r^{(1/2)}(x^2).$$

Results (4.3) and (4.4) have also been given by Szegö [1], the proof being completely different.

**5. Another generating relation.** From (3.1), we get:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(C)_n H_{n,m,\nu}(x)t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (C)_n (\nu x)^{n-mk} t^n}{k!(n-mk)!} \\ &= \sum_{n,k=0}^{\infty} \frac{(-1)^k (C)_{mk} (c+mk)_n (\nu x)^{n} t^{n+mk}}{k! n!} \\ &= \sum_{k=0}^{\infty} {}_1F_0[c+mk; -; \nu xt] \frac{m^{mk} \Delta_k(m; c)(-t^m)^k}{k!} \\ &= (1-\nu xt)^{-c} {}_mF_0 \left[ \Delta(m; c); -; - \left( \frac{mt}{1-\nu xt} \right)^m \right]. \end{aligned}$$

We thus arrive at the divergent generating relation:

$$(5.1) \quad \begin{aligned} (1-\nu xt)^{-c} {}_mF_0 \left[ \Delta(m; c); -; - \left( \frac{mt}{1-\nu xt} \right)^m \right] \\ \cong \sum_{n=0}^{\infty} \frac{(c)_n H_{n,m,\nu}(x)t^n}{n!}. \end{aligned}$$

*Particular case.* when  $m=\nu=2$ , we get the well-known result:

$$(1-2xt)^{-c} {}_2F_0 \left[ \Delta(2; c); -; - \left( \frac{2t}{1-2xt} \right)^2 \right] \cong \sum_{n=0}^{\infty} \frac{(c)_n H_n(x)t^n}{n!}.$$

## 6. *n*th differential formulae.

*First form.* From (3.1), we get:

$$\begin{aligned}
 \frac{H_{n,m,\nu}(x)}{n!} &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (\nu x)^{n-mk}}{k!(n-mk)!} \\
 &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k \{ D^{(m-1)n} x^{mn-mk} \} \nu^{n-mk}}{k!(mn-mk)!} \\
 (6.1) \quad &= D^{(m-1)n} \sum_{k=0}^n \frac{(-1)^k \nu^{n-mk} x^{mn-mk}}{k!(mn-mk)!}
 \end{aligned}$$

because  $D^{(m-1)n} x^{mn-mk} = 0$ , for  $\lfloor n/m \rfloor < k \leq n$ .

$$\begin{aligned}
 \therefore H_{n,m,\nu}(x) &= D^{(m-1)n} n! \sum_{k=0}^n \frac{(-1)^{n-k} \nu^{n-m(n-k)} x^{mk}}{(n-k)!(mk)!} \\
 &= (-1)^n \nu^{(1-m)n} D^{(m-1)n} \sum_{k=0}^n \frac{(-n)_k (\nu x)^{mk}}{k! m^{mk} \Delta_k[m(1); 1]} \\
 (6.2) \quad &= (-1)^n \nu^{(1-m)n} D^{(m-1)n} {}_1F_{m-1} \left[ -n; \Delta(m(1); 1); \left( \frac{\nu x}{m} \right)^m \right].
 \end{aligned}$$

*Particular cases.* (i) When  $m=\nu=2$  in (6.2), we get

$$(6.3) \quad H_n(x) = \frac{n!}{\left(\frac{1}{2}\right)_n} \left(-\frac{1}{2}\right)^n D^n L_n^{(-1/2)}(x^2).$$

(ii) For  $n=2r$ ,  $m=\nu=2$  and with the help of (4.3) we get:

$$(6.4) \quad L_r^{(-1/2)}(x^2) = \frac{(-1)^r \left(\frac{1}{2}\right)_r}{2^{2r} \left(\frac{1}{2}\right)_{2r}} D^{2r} L_{2r}^{(-1/2)}(x^2).$$

(iii) Again for  $n=2r+1$ ,  $m=\nu=2$  and with the help of (4.4), we get:

$$(6.5) \quad L_r^{(1/2)}(x^2) = \frac{(-1)^{r+1} (2r+1) \left(\frac{1}{2}\right)_r}{2^{2r+2} \left(\frac{1}{2}\right)_{2r+1} x} D^{2r+1} L_{2r+1}^{(-1/2)}(x^2).$$

*Second form.* From (6.1), we get:

$$(6.6) \quad H_{n,m,\nu}(x) = D^{(m-1)n} n! \sum_{k=0}^n \frac{(-1)^{(m+1)k} \nu^{n-mk} (-mn)_{mk} x^{mn-mk}}{k!(mn)!}$$

$$\begin{aligned}
 (6.7) \quad &= \frac{\nu^n n!}{(mn)!} D^{(m-1)n} x^{mn} \sum_{k=0}^n \frac{m^{mk} \Delta_k(m; -mn) (-1)^{(m+1)k}}{k! (\nu x)^{mk}} \\
 &= \frac{\nu^n n!}{(mn)!} D^{(m-1)n} \left\{ x^{mn} {}_m F_0 \left[ \Delta(m; -mn); -; - \left( -\frac{m}{\nu x} \right)^m \right] \right\}.
 \end{aligned}$$

*Particular cases.* (i) When  $m = \nu = 2$ , we get:

$$(6.8) \quad H_n(x) = \frac{n!}{2^n(2n)!} D^n H_{2n}(x).$$

(ii) When  $n = 2r$  and  $m = \nu = 2$ , from (4.3), we get

$$(6.9) \quad L_r^{(-1/2)}(x^2) = \frac{(-1)^r (\frac{1}{2})_r}{2^{2r}(4r)!} D^{2r}[H_{4r}(x)].$$

(iii) When  $n = 2r+1$  and  $m = \nu = 2$ , from (4.4), we have

$$(6.10) \quad L_r^{(1/2)}(x^2) = \frac{(-1)^r}{r! 2^{8r+4} (\frac{1}{2})_{2r+1} x} D^{2r+1}[H_{4r+2}(x)].$$

(iv) From (6.9) and (4.3), we get

$$(6.11) \quad L_r^{(-1/2)}(x^2) = \frac{(-1)^r (\frac{1}{2})_r}{2^{2r} (\frac{1}{2})_{2r}} D^{2r} L_{2r}^{(-1/2)}(x^2).$$

(v) From (6.9) and (4.3), we get

$$(6.12) \quad L_r^{(1/2)}(x^2) = \frac{(-1)^{r+1} (2r+1) (\frac{1}{2})_r}{2^{2r+2} (\frac{1}{2})_{2r+1} x} D^{2r+1} L_{2r+1}^{(-1/2)}(x^2).$$

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#### REFERENCE

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