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ON THE INVERTIBILITY OF GENERAL WIENER-HOPF OPERATORS

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ABSTRACT. Let $\mathfrak{H}$ be a separable Hilbert space, $\mathfrak{B}$ the set of bounded linear operators on $\mathfrak{H}$, and P an orthogonal projection on $\mathfrak{H}$. Denote the range of P by $R(P)$. Let A belong to $\mathfrak{B}$. The *general Wiener-Hopf operator* associated with A and P is defined by $T_P(A)=PA|R(P)$, the vertical bar denoting restriction. Let $Q=I-P$. The purpose of this paper is to disprove the general conjecture that *if A is an invertible element of $\mathfrak{B}$, then the invertibility of $T_P(A)$ implies the invertibility of $T_Q(A)$.* We also disprove the conjecture in an interesting special case.

1. Introduction. Let $\mathfrak{H}$ be a separable Hilbert space, and $\mathfrak{B}$ the set of bounded linear operators on $\mathfrak{H}$. Let P be an orthogonal projection on $\mathfrak{H}$. Denote its range by $R(P)$. Let A belong to $\mathfrak{B}$. The *general Wiener-Hopf operator* associated with A and P is defined by

$$T_P(A) = PA \mid R(P),$$

the vertical bar denoting restriction. Let $Q=I-P$. The purpose of this paper is to disprove the general conjecture that *if A is an invertible element of $\mathfrak{B}$, then the invertibility of $T_P(A)$ implies the invertibility of $T_Q(A)$.* We also disprove the conjecture in an interesting special case.

In §2 we mention some special cases in which the conjecture has been proven true. We then exhibit a simple counterexample to prove that the conjecture is, in general, false.

In order to discuss §3 we introduce some additional terminology. Let $L^2(T; \mathfrak{H})$ denote the space of equivalence classes of $\mathfrak{H}$-valued, weakly measurable functions on the circle group T which are square summable; $H^2(T; \mathfrak{H})$ the subspace of $L^2(T; \mathfrak{H})$ which consists of those elements whose Fourier coefficients vanish on the negative integers; and $K^2(T; \mathfrak{H})$ the orthogonal complement of $H^2(T; \mathfrak{H})$ in $L^2(T; \mathfrak{H})$. P and Q are projections from $L^2(T; \mathfrak{H})$ onto $H^2(T; \mathfrak{H})$ and $K^2(T; \mathfrak{H})$ respectively. *All operators are assumed to be bounded and linear.* Let $A(\theta)$ be any weakly measurable, essentially bounded, $\mathfrak{B}$-valued function on the circle group. We define an operator A on $L^2(T; \mathfrak{H})$ by

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$$(Af)(\theta) = A(\theta)f(\theta), \quad f \in L^2(T; \mathfrak{H}), \quad \theta \in T,$$

and two Wiener-Hopf operators by

$$T_P(A) = PA \mid H^2(T; \mathfrak{H}), \qquad T_Q(A) = QA \mid K^2(T; \mathfrak{H}).$$

As usual, $\mathfrak{C}$ denotes the complex numbers. Finally, suppose that the operator A is invertible.

Devinatz and Shinbrot [1] have shown that if H is one-dimensional, then $T_P(A)$ being invertible implies $T_Q(A)$ is invertible. They remark: "*Whether or not this result persists when the dimension of* $\mathfrak{H}$ *is greater than* 1 *is not known.*" (The remark follows proof of Corollary 8, §6 in [1].) We shall prove that the result does not by exhibiting a counterexample when the dimension of $\mathfrak{H}$ is two (i.e. $\mathfrak{H} = \mathfrak{C} \times \mathfrak{C}$). The counterexample is of the form U^*BU where U is a certain unitary operator on $L^2(T; \mathfrak{C}\times\mathfrak{C})$ mapping $H^2(T; \mathfrak{C}\times\mathfrak{C})$ onto $L^2(T; \mathfrak{C}) \times \{0\}$ and $K^2(T; \mathfrak{C}\times\mathfrak{C})$ onto $\{0\} \times L^2(T; \mathfrak{C})$, and B is an invertible operator on $L^2(T; \mathfrak{C}\times\mathfrak{C})$ patterned after the counterexample of §2. This counterexample may be generalized to disprove the result when the dimension of $\mathfrak{H}$ is *any* integer greater than one.

2. **The first counterexample.** Devinatz and Shinbrot have proven that for a unitary operator U, the invertibility of $T_P(U)$ is equivalent to that of $T_Q(U)$ for any orthogonal projection P on $\mathfrak{H}$ where $Q = I - P$ (Corollary 1, §2 in [1]). They have also shown that if A is any operator with a strongly positive real part ($\mathrm{Re}(Ax, x) \geqq \delta(x, x)$, $x \in \mathfrak{H}$, δ positive), then $T_P(A)$ is invertible for every orthogonal projection P. In particular, it is invertible for $Q = I - P$ (Lemma 2, §2 in [1]). Pellegrini proved that for an operator A, $T_P(A)$ is invertible for every orthogonal projection if and only if there exists a θ between 0 and 2π such that $e^{i\theta}A$ has strongly positive real part (Theorem 1.2.10 in [2]). The following simple counterexample illustrates that the invertibility of $T_P(A)$ does not guarantee that of $T_Q(A)$, even if A is invertible.

COUNTEREXAMPLE 1. Take $\mathfrak{H} = \mathfrak{C} \times \mathfrak{C}$, P the projection from $\mathfrak{H}$ onto $\mathfrak{C} \times \{0\}$, and $Q = I - P$. Let A be the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The determinant of A is -1 so that A is invertible. It is easily seen that $T_P(A) = I \mid \mathfrak{C} \times \{0\}$ and $T_Q(A) = 0 \mid \{0\} \times \mathfrak{C}$.

3. **The second counterexample.** In this section P is the projection from $L^2(T; \mathfrak{C}\times\mathfrak{C})$ onto $H^2(T; \mathfrak{C}\times\mathfrak{C})$ and $Q = I - P$; P' is the projection from $L^2(T; \mathfrak{C}\times\mathfrak{C})$ onto $L^2(T, \mathfrak{C}) \times \{0\}$ and $Q' = I - P'$. We shall construct an invertible operator A on $L^2(T; \mathfrak{C}\times\mathfrak{C})$ such that $T_P(A)$ is invertible but $T_Q(A)$ is not.

Let $[\ ,\ ]$ denote the inner product in both $\mathfrak{C}\times\mathfrak{C}$ and $\mathfrak{C}$; $\langle\ ,\ \rangle$ the inner product in both $L^2(T;\mathfrak{C}\times\mathfrak{C})$ and $L^2(T;\mathfrak{C})$. If $f(\theta)\in L^2(T;\mathfrak{C}\times\mathfrak{C})$, then f is easily seen to be of the form $(f_1(\theta), f_2(\theta))$ where $f_1, f_2\in L^2(T;\mathfrak{C})$. Let $g(\theta)=(g_1(\theta), g_2(\theta))$ be another element of $L^2(T;\mathfrak{C}\times\mathfrak{C})$. The topology on $L^2(T;\mathfrak{C}\times\mathfrak{C})$ is determined by the inner product:

$$\langle f, g\rangle = \int_{\theta\in T}[f(\theta), g(\theta)]d\theta = \int_{\theta\in T}[(f_1(\theta), f_2(\theta)), (g_1(\theta), g_2(\theta))]d\theta. \tag{3.1}$$

We define inner product on $\mathfrak{C}\times\mathfrak{C}$ to be the sum of the respective inner products. Thus, (3.1) becomes

$$\begin{aligned}\langle f, g\rangle &= \int_{\theta\in T}[f_1(\theta), g_1(\theta)]d\theta + \int_{\theta\in T}[f_2(\theta), g_2(\theta)]d\theta \\ &= \langle f_1, g_1\rangle + \langle f_2, g_2\rangle.\end{aligned}\tag{3.2}$$

This indicates that we may identify $L^2(T;\mathfrak{C}\times\mathfrak{C})$ with $L^2(T;\mathfrak{C})\times L^2(T;\mathfrak{C})$ where the inner product in the latter space is given by the right-hand side of (3.2). Under this identification, $H^2(T;\mathfrak{C}\times\mathfrak{C})$ and $K^2(T;\mathfrak{C}\times\mathfrak{C})$ become $H^2(T;\mathfrak{C})\times H^2(T;\mathfrak{C})$ and $K^2(T;\mathfrak{C})\times K^2(T;\mathfrak{C})$ respectively. We make this identification freely throughout the remainder of the paper.

The Plancherel theorem says that the Fourier transform, F, is an isometry from $L^2(T;\mathfrak{C})$ onto l^2, the square-summable, $\mathfrak{C}$-valued sequences on the integers. The adjoint of F, F^*, is easily seen to equal F^{-1}. We make use of these facts in the following lemma.

LEMMA 1. *There exists a unitary operator U on $L^2(T;\mathfrak{C}\times\mathfrak{C})$ such that*

(i) $U=U^*=U^{-1}$,
(ii) $U(H^2(T;\mathfrak{C}\times\mathfrak{C}))=L^2(T;\mathfrak{C})\times\{0\}$,
(iii) $U(K^2(T;\mathfrak{C}\times\mathfrak{C}))=\{0\}\times L^2(T;\mathfrak{C})$.

PROOF. For $(f_n), (g_n)\in l^2$, define a map $G: l^2\times l^2\to l^2\times l^2$ by

$$G((f_n), (g_n)) = ((h_n), (k_n))$$

where

$$\begin{aligned} h_n &= f_n, && n\geqq 0,\\ &= g_{-n-1}, && n<0,\end{aligned}$$

$$\begin{aligned} k_n &= f_{-n-1}, && n\geqq 0,\\ &= g_n, && n<0.\end{aligned}$$

$$\|G((f_n),(g_n))\|^2 = \sum f_n\bar{f}_n + \sum g_{-n-1}\bar{g}_{-n-1} + \sum f_{-n-1}\bar{f}_{-n-1} + \sum g_n\bar{g}_n$$
$$= \|(f_n)\|^2 + \|(g_n)\|^2.$$

The inner product in $l^2 \times l^2$ is taken to be the sum of the respective inner products in l^2, so that $\|((f_n),(g_n))\|^2 = \|(f_n)\|^2 + \|(g_n)\|^2$. Observe that G is linear. Hence G is an isometry. Note that $G^2 = I$, so that $G = G^{-1}$. Let $h^2 \times h^2 (k^2 \times k^2)$ denote the subspace of $l^2 \times l^2$ of series whose terms vanish for negative (positive) indices. Observe that G maps $h^2 \times h^2$ onto $l^2 \times \{0\}$ and $k^2 \times k^2$ onto $\{0\} \times l^2$.

Now we define a map $H: L^2(T;\ \mathfrak{C}) \times L^2(T;\ \mathfrak{C}) \to l^2 \times l^2$ via the Fourier transform, F, as follows

$$H(f,g) = (F(f), F(g)), \qquad (f,g) \in L^2(T;\mathfrak{C}) \times L^2(T;\mathfrak{C}),$$
$$\|H(f,g)\|^2 = \|F(f)\|^2 + \|F(g)\|^2 = \|f\|^2 + \|g\|^2.$$

Note that H is linear because F is linear. Hence H is an isometry. It takes $H^2(T;\ \mathfrak{C}) \times H^2(T;\ \mathfrak{C})$ onto $h^2 \times h^2$, $K^2(T;\ \mathfrak{C}) \times K^2(T;\ \mathfrak{C})$ onto $k^2 \times k^2$, $L^2(T;\ \mathfrak{C}) \times \{0\}$ onto $l^2 \times \{0\}$, and $\{0\} \times L^2(T;\ \mathfrak{C})$ onto $\{0\} \times l^2$. It is easily seen that $H^* = H^{-1}$.

Finally, we set $U = H^{-1}GH$. If we make the identification of $L^2(T;\ \mathfrak{C} \times \mathfrak{C})$ with $L^2(T;\ \mathfrak{C}) \times L^2(T;\ \mathfrak{C})$ mentioned earlier, then U may be considered as an operator taking $L^2(T;\ \mathfrak{C} \times \mathfrak{C})$ onto itself. Now (ii) and (iii) follow immediately from the above mentioned properties of G and H. We prove (i):

$$U^* = H^*G^*(H^{-1})^* = H^{-1}GH = U, \quad U^{-1} = H^{-1}G^{-1}H = H^{-1}GH = U. \blacksquare$$

The following operator is central to the forthcoming counterexample. Its construction is patterned after that of Counterexample 1. Let

$$B(\theta) = \begin{pmatrix} b_{11}(\theta) & b_{12}(\theta) \\ b_{21}(\theta) & b_{22}(\theta) \end{pmatrix} \tag{3.3}$$
$$\text{where} \quad \begin{cases} b_{11}(\theta) = b_{12}(\theta) = b_{21}(\theta) = 1, \\ b_{22}(\theta) = 0, \end{cases} \qquad \theta \in T.$$

Define

$$(Bf)(\theta) = B(\theta)f(\theta), \qquad f \in L^2(T;\mathfrak{C} \times \mathfrak{C}), \quad \theta \in T. \tag{3.4}$$

The properties of B which will be of interest to us are contained in the following lemma.

LEMMA 2. *B is an invertible operator on $L^2(T;\ \mathfrak{C} \times \mathfrak{C})$ such that*
(i) $T_{P'}(B) = I \mid L^2(T;\mathfrak{C}) \times \{0\}$,
(ii) $T_{Q'}(B) = 0 \mid \{0\} \times L^2(T;\mathfrak{C})$.

PROOF. For any $f \in L^2(T;\ \mathfrak{C}\times\mathfrak{C})$ we have shown that $f=(f_1, f_2)$ where $f_1, f_2 \in L^2(T;\ \mathfrak{C})$. By (3.3), $B(f_1, f_2)=(f_1+f_2, f_1)$. Hence (i) and (ii) follow immediately. It is easily seen that $B^{-1}(f_1, f_2)=(f_2, f_1-f_2)$. The fact that B and B^{-1} are bounded and linear is also easily shown.■

The following simple lemma is proven in a general context.

LEMMA 3. *Let U be a unitary operator on a Hilbert space $\mathfrak{H}$. Suppose P and P' are two orthogonal projections on $\mathfrak{H}$ such that $U(R(P)) =R(P')$ and $U(R(Q))=R(Q')$ $(Q=I-P,\ Q'=I-P')$. Let B belong to $\mathfrak{B}$. Then*

(i) $T_P(U^*BU)=U^*T_{P'}(B)U \mid R(P)$,

(ii) $T_Q(U^*BU)=U^*T_{Q'}(B)U \mid R(Q)$.

PROOF. First we note that $(U^*P'U)^2=U^*P'U$, $(U^*P'U)^*=U^*P'U$, and $R(U^*P'U)=R(P)$. Hence $P=U^*P'U$. The rest is easy:

$$\begin{aligned} U^*T_{P'}(B)U \mid R(P) &= U^*(P'UU^*B \mid R(P'))U \mid R(P) \\ &= P(U^*BU) \mid R(P) = T_P(U^*BU). \end{aligned}$$

The proof for $T_Q(U^*BU)$ is identical.■

We are now prepared to exhibit the following counterexample.

COUNTEREXAMPLE 2. *There exists an invertible operator A on $L^2(T;\ \mathfrak{C}\times\mathfrak{C})$ such that $T_P(A)=I \mid H^2(T;\ \mathfrak{C}\times\mathfrak{C})$ and $T_Q(A)= 0 \mid K^2(T;\mathfrak{C}\times\mathfrak{C})$.*

PROOF. Let U be the operator of Lemma 1; B, that of Lemma 2; and set $A=U^*BU$. By Lemmas 1 and 2, A is an invertible operator on $L^2(T;\mathfrak{C}\times\mathfrak{C})$. Moreover, Lemma 3 says that

$$U^*T_{P'}(B)U \mid H^2(T;\mathfrak{C}\times\mathfrak{C}) = T_P(U^*BU).$$

But $T_{P'}(B)=I \mid L^2(T;\ \mathfrak{C})\times\{0\}$ by (i) of Lemma 2. Thus, using (i) and (ii) of Lemma 1, we get that

$$T_P(A) = T_P(U^*BU) = I \mid H^2(T;\mathfrak{C}\times\mathfrak{C}).$$

Similarly, one sees that

$$T_Q(A) = T_Q(U^*BU) = U^*T_{Q'}(B)U \mid K^2(T;\mathfrak{C}\times\mathfrak{C}) = 0 \mid K^2(T;\mathfrak{C}\times\mathfrak{C}).■$$

REFERENCES

1. A. Devinatz and M. Shinbrot, *General Wiener-Hopf operators*, Trans. Amer. Math. Soc. **145** (1969), 467–494.

2. V. J. Pellegrini, *Wiener-Hopf operators*, Ph.D. Thesis, Northwestern University, Evanston, Ill.

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