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FIXED POINTS FOR CONTRACTIVE MULTIFUNCTIONS

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ABSTRACT. Let $F:X\to X$ be a point closed multifunction on the bounded metric space (X, d). Let $\hat{d}$ denote the Hausdorff metric for the nonempty closed subsets of X. Then F is contractive iff $\hat{d}(F(x), F(y))<d(x, y)$ for all $x, y\in X$. We give conditions under which contractive multifunctions have fixed points.

1. **Introduction.** The fixed point theorem for contraction maps on a complete metric space into itself is well known, and a number of generalizations of this result have appeared [1], [2], [3]. Further, Nadler [5] has proved an extension of the fixed point theorem for contraction maps to multivalued functions. The purpose of the present paper is to extend a fixed point theorem of Edelstein's [2] for contractive mappings to multivalued functions. In the proof of the main theorem, we make use of Edelstein's methods.

In the following X will denote a bounded metric space with metric d, and we shall let $\hat{d}$ denote the Hausdorff metric on the space $S(X)$ of nonempty closed subsets of X. A *multifunction* $F:X\to X$ is a point to set correspondence (i.e. a multivalued function), and we use upper case F, G, etc. to denote a multifunction. Further, the multifunction is called *point closed* (*compact*) in case $F(x)$ is a closed (compact) set for all $x\in X$. If $A\subset X$, then $F(A)=\bigcup\{F(x):x\in A\}$ and $F^{-1}(A) =\{x:F(x)\cap A\neq\emptyset\}$. We shall use the following definitions.

DEFINITION. The multifunction F is upper semicontinuous (U.S.C.) iff for each closed set $A\subset X$, the set $F^{-1}(A)$ is closed.

One of our major tools will be the concept of an orbit of the multifunction F at the point x.

DEFINITION. An orbit $\mathcal{O}(x)$ of the multifunction F at the point x is a sequence $\{x_n:x_n\in F(x_{n-1})\}$ where $x_0=x$. We shall use $\mathcal{O}(x)$ as a sequence and as a set as the situation demands.

2. **The main theorem.** Before stating the main theorem of this paper, we need some more definitions and preliminary results.

DEFINITION. The multifunction F is *contractive* iff for each $x_1, x_2\in X$ with $x_1\neq x_2$, $\hat{d}(F(x_1), F(x_2))<d(x_1, x_2)$.

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An immediate consequence of the definition is: If $y_1 \in F(x_1)$, then there is an element $y_2 \in F(x_2)$ such that $d(y_1, y_2) < d(x_1, x_2)$.

DEFINITION. An orbit $\mathcal{O}(x)$ is called a *regular orbit* iff

$$d(x_{n+1}, x_{n+2}) \leqq d(x_n, x_{n+1}) \quad \text{and} \quad d(x_{n+1}, x_{n+2}) \leqq \hat{d}(F(x_n), F(x_{n+1})).$$

REMARKS. (1) Let F be a point compact, contractive multifunction. Define an orbit $\mathcal{O}(x)$ by choosing $x_n \in F(x_{n-1})$ so that

$$d(x_{n-1}, x_n) = d(x_{n-1}, F(x_{n-1})) = \inf\{d(x_{n-1}, y) : y \in F(x_{n-1})\}.$$

Then, since F is contractive, it follows that $\mathcal{O}(x)$ is regular.

(2) It is fairly easy to construct simple examples to show that a contractive multifunction need not be U.S.C. However, if F is point compact and contractive, then one can show that F is U.S.C.

Furthermore, if F is a point compact U.S.C. multifunction, if $x_n \to x_0$, and if $y_n \in F(x_n)$ for each n, then one can show that there is a subsequence $\{y_{n_i} : i \geqq 1\}$ which converges to a point in $F(x_0)$. Similarly one can show that if F is point closed and contractive, if $x_n \to x_0$ and if $y_n \to y_0$ with $y_n \in F(x_n)$, then $y_0 \in F(x_0)$.

We are now ready to state our main theorem.

THEOREM 1. *Let F be a point closed, contractive multifunction. If there is a regular orbit $\mathcal{O}(x)$ for F which contains a convergent subsequence $x_{n_i} \to y_0$ such that $x_{n_i+1} \to y_1$, then $y_1 = y_0$ (i.e. F has a fixed point).*

PROOF. Let $\mathcal{O}(x)$ be a regular orbit with $x_{n_i} \to y_0$, $x_{n_i+1} \to y_1$ and $y_1 \in F(y_0)$. We define a function $f: Y = X \times X \setminus \Delta \to R$ where Δ is the diagonal as follows and R is the reals:

$$r(p, q) = \hat{d}(F(p), F(q))/d(p, q).$$

Then r is a continuous function and since F is contractive, $r(p, q) < 1$. Thus if $y_1 \neq y_0$ there is an a, $0 < a < 1$, and an open set U of Y such that $(y_0, y_1) \in U$ and if $(p, q) \in U$, then $0 \leqq r(p, q) < a$. Now choose $\rho > 0$ so that (i) $\rho < \frac{1}{3} d(y_0, y_1)$ and (ii) if $B_1 = B(y_0, \rho)$, $B_2 = B(y_1, \rho)$ then $B_1 \times B_2 \subset U$.

Since $x_{n_i} \to y_0$ and $x_{n_i+1} \to y_1$, there is an N such that $i \geqq N$ implies that $x_{n_i} \in B_1$ and $x_{n_i+1} \in B_2$. Therefore $d(x_{n_i}, x_{n_i+1}) > \rho$ for all $i > N$.

On the other hand from the definition of r and the choice of U

$$\hat{d}(F(x_{n_i}), F(x_{n_i+1})) < a d(x_{n_i}, x_{n_i+1}),$$

and since $\mathcal{O}(x)$ is regular, we get $d(x_{n_i+1}, x_{n_i+2}) < a\, d(x_{n_i}, x_{n_i+1})$. Further, if $l > j > N$,

$$d(x_{n_l}, x_{n_l+1}) \leqq d(x_{n_{l-1}+1}, x_{n_{l-1}+2}) < ad(x_{n_{l-1}}, x_{n_{l-1}+1}).$$

Then by repeating this argument we get: $d(x_{n_l}, x_{n_l+1}) < a^{l-j}d(x_{n_j}, x_{n_j+1})$. But with fixed j, $a^{l-j} \to 0$ as $l \to \infty$, which implies that $d(x_{n_l}, x_{n_l+1}) \to 0$ as $l \to \infty$. This contradicts $d(x_{n_l}, x_{n_l+1}) > \rho$ for $l > N$. Thus we conclude that $y_0 = y_1$ and hence, F has a fixed point.

We can deduce a theorem of Fraser and Nadler [4] from Theorem 1. For this let F be a multifunction and define $\hat{F}$ by $\hat{F}(A) = \bigcup \{F(x) : x \in A\}$ for $A \subset X$. If F is point compact and contractive, then $\hat{F}$ is a continuous function on the compact subsets of X into the compact subsets of X. Then we get

COROLLARY 1.1. *Let F be a point compact, contractive multifunction on X. If there is a compact subset A of X such that some subsequence of the sequence $\{\hat{F}^n(A) : n \geqq 1\}$ of iterates of $\hat{F}$ at A converges to a compact set, then $\hat{F}$ has a fixed point.*

Finally, another corollary is:

COROLLARY 1.2. *If F is a point closed, contractive multifunction on the compact, metric space X into itself, then F has a fixed point.*

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