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A NEW PROOF OF BROWN'S COLLARING THEOREM

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The aim of this note is to give a new proof that if a subspace B, compact for convenience, is locally collared in a space X, then it is collared. The idea of the proof is simply to add a collar $B\times I$ to X to get X^+ and then to construct a homeomorphism of X with X^+ by pushing B down on one collared open set at a time.

The theorem, of course, is essentially that of [1]. However, the proof easily works in the piecewise linear (PL) category (i.e. all maps are PL and spaces are polyhedra), and when B, the boundary, is a pair or flag, cf. [3]. At the end of the paper we shall note briefly how the noncompact case and the PL case can be handled by our techniques.

A closed subspace $B\subset X$ is *locally collared* if B is covered by sets U, open in B, such that for each U there is a closed embedding $h:\overline{U}\times[0, 1]\to X$ such that $h^{-1}(B)=\overline{U}\times\{0\}$, $h(x, 0)=x$ for $x\in\overline{U}$, and $h(U\times[0, 1))$ is open in X. For metric spaces this is equivalent to the definition in [1]. B is said to be *collared* if one U can be taken to be all of B.

THEOREM. *If $B\subset X$ is compact and locally collared in X, which is Hausdorff, then B is collared in X.*

PROOF. Let $U_1, U_2, \cdots, U_n$ be an open cover of B such that each U_i is as in the definition. By the normality of B, shrink the cover to find another cover $V_1, \cdots, V_n$ such that $\overline{V}_i\subset U_i$, $i=1, \cdots, n$. Let $X^+=X\cup B\times[-1, 0]$ where $(x, 0)$ is identified to x, and let $h_i:\overline{U}_i\times[0, 1]\to X$ be the embeddings given by the local collars. Let $H_i:\overline{U}_i\times[-1, 1]\to X^+$, $i=1, \cdots, n$, be the embedding defined by

$$\begin{aligned} H_i(x) &= h_i(x) \quad \text{for } x\in\overline{U}_i\times[0,1],\\ &= x \qquad\quad \text{for } x\in\overline{U}_i\times[-1,0]. \end{aligned}$$

Inductively we shall define maps $f_i:B\to[-1, 0]$ and embeddings $g_i:X\to X^+$, $i=0, 1, \cdots, n$, such that

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(a) $f_i(x) = -1$ if $x \in \bigcup_{j \leq i} \overline{V}_j$,
(b) $g_i(x) = (x, f_i(x))$ if $x \in B$, and
(c) $g_i(X) = X \cup \{(x, t) \mid t \geqq f_i(x)\}$.
Note that since the V_i's cover B, $g_n(X) = X^+$ and thus g_n^{-1} will give the required collar.

Define $g_0 = 1$, and inductively suppose g_{i-1} has been defined. Let $\phi_i: H_i^{-1}g_{i-1}(X) \to \overline{U}_i \times [-1, 1]$ be an embedding that pushes down along fibers such that $\phi_i H_i^{-1} g_{i-1}(\overline{V}_i) = \overline{V}_i \times \{-1\}$ and $\phi_i | (\overline{U}_i - U_i) \times [-1, 1] \cup \overline{U}_i \times \{1\} = 1$. Such a ϕ_i can be defined as follows: Let $\lambda_i: \overline{U}_i \to [0, 1]$ be a Urysohn function which is 0 on $\overline{U}_i - U_i$ and 1 on $\overline{V}_i$. Let $s_x: [f_{i-1}(x), 1] \to [(1-\lambda_i(x))f_{i-1}(x) + \lambda_i(x)(-1), 1]$ be the unique order preserving simplicial homeomorphism given by $s_x(t) = ((b-1)/(a-1))(t-1)+1$ where $a = f_{i-1}(x)$ and $b = (1-\lambda_i(x))f_{i-1}(x) + \lambda_i(x)(-1)$. Now define $\phi_i(x, t) = (x, s_x(t))$. Clearly ϕ_i is continuous. Then define $\Phi_i: g_{i-1}(X) \to X^+$ by:

$$\begin{aligned} \Phi_i(x) &= H_i \phi_i H_i^{-1}(x) \quad \text{for } x \in g_{i-1}(X) \cap H_i(\overline{U} \times [-1, 1]), \\ &= x \qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

and $g_i = \Phi_i g_{i-1}$. Clearly Φ_i and thus g_i is well defined and an embedding since $\phi_i | (\overline{U}_i - U_i) \times [-1, 1] \cup \overline{U}_i \times \{1\} = 1$, ϕ_i is an embedding (since each s_x is), and $g_{i-1}(X) \cap H_i(\overline{U}_i \times [-1, 1]) = H_i(\overline{U}_i \times [0, 1]) \cup \{(x, t) \mid t \geqq f_{i-1}(x) \text{ and } x \in \overline{U}_i\}$ by (c) for g_{i-1}. Note that (b) now defines $f_i(x)$, and (a) and (c) are satisfied by construction.

REMARK 1. *The noncompact case.* The method of proof used above can be extended to certain cases when X is not compact. For instance, the proof works if we assume that X has a slightly stronger property than paracompactness, namely if every open cover has a star finite refinement (cf. [2] for definitions). In this case it is possible to order the U_i's, although infinite, so that every point in X^+ has an open neighborhood which moves only finitely often.

REMARK 2. *The* PL *case.* The theorem is still true if all spaces and maps mentioned (including the definition of local collaring) are polyhedra and PL respectively. The same proof goes through except that the particular definition of ϕ_i must be altered slightly. Namely to make ϕ_i PL it is easiest to triangulate $\overline{U}_i \times [-1, 1]$ so that $\overline{V}_i \times [-1, 1]$ and $H_i^{-1}\, g_{i-1}(X)$are subcomplexes and projection $\pi: \overline{U}_i \times [-1, 1] \to \overline{U}_i$ is simplicial. Then it is easy to define a simplicial map ϕ_i so that it has the desired properties.

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