PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, January 1971

INERTIAL AND BORDISM PROPERTIES OF SPHERES

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ABSTRACT. The k-connective bounding group $\theta^n(k)$ and the k-connective inertial group $I^n(k)$ are defined as subgroups of θ^n, the group of smooth n-spheres, $n \geqq 7$. It is shown $I^n(k)$ is contained in $\theta^n(k)$. Consequently, the image of the Milnor-Novikov pairing $\tau_{n,k}$ is contained in $\theta^{n+k}(k)$ when $n \geqq k+2$. It follows that $\tau_{7,3}=0$.

1. **Introduction.** Let θ^n be the group of oriented diffeomorphism classes of manifolds homeomorphic to the usual n-sphere, S^n. Assume $n \geqq 7$. Define $\theta^n(k)$ to be the subgroup of θ^n consisting of those Σ^n which are the boundaries of k-connected $(n+1)$-dimensional compact manifolds, $1 \leqq k < [n/2]$. Thus, $\theta^n(k)$ is the kernel of the natural map $i_k:\theta^n \to \Omega_n(k)$ where $\Omega_n(k)$ is the n-dimensional group in k-connective cobordism theory [10] and i_k sends $\Sigma^n \in \theta^n$ to its cobordism class. Using surgery, we see $\Omega_*(1)$ is the usual oriented cobordism group so $\theta^n = \theta^n(1)$. Similarly, $\Omega_n(2) \approx \Omega_n^{\text{spin}}$ (recall $n \geqq 7$); since BSpin is, in fact, 3-connected, for $n \geqq 8$ $\Omega_n(2) \approx \Omega_n(3)$ and $\theta^n(2) = \theta^n(3) = b\mathrm{Spin}_n$. Similarly, $\theta^n(k) = \theta^n(k+1)$ for $n \geqq 2k+4$ and $k \equiv 2, 4, 5, 6 \pmod 8$. θ^n is filtered

$$\theta^n = \theta^n(1) \supset \theta^n(2) \supset \cdots \supset \theta^n([n/2]-1) \supset \theta^n(\partial\pi).$$

The last inclusion is demonstrated in [5].

Let $I^n(k)$ be the group of those $\Sigma^n \in \theta^n$ such that for some k-connected closed manifold M^n, $M^n \# \Sigma^n$ is diffeomorphic to M^n. All diffeomorphisms are assumed to preserve orientation. In the notation of [6], $\Sigma \in I(M)$, the inertia group of M. Our main result is

THEOREM. *$I^n(k)$ is a subgroup of $\theta^n(k)$.*

We postpone the proof to the next section. Wall [13] has proved this result when $n=2m$, $k=m-1$ with a different construction. In general, the groups of the theorem are not equal. This is discussed in §3.

Let $\tau_{n,k}:\theta^n \otimes \pi_k(SO(n-1)) \to \theta^{n+k}$ be the Milnor-Munkres-Novikov pairing [7, p. 583], [4]. Munkres has shown [7, p. 577], [9] that if

Received by the editors October 17, 1969.

AMS 1969 *subject classifications.* Primary 5710.

Key words and phrases. k-connective cobordism, inertia group, exotic spheres.

[1] This research was supported in part by an I.I.T. Faculty Research Fellowship and by NSF contract GP-9614.

M^{n+k} contains an embedded S^{k+1} with normal bundle having characteristic class $\alpha \in \pi_k(SO(n-1))$, then for any $\Sigma' \in \theta^n$, $\tau_{n,k}(\Sigma' \otimes \alpha) \in I(M)$. In particular, if M^{n+k} is the total space of the S^{n-1}-bundle over S^{k+1} with characteristic class $S\alpha$, $\alpha \in \pi_k(SO(n-2))$, $\tau_{n,k}(\theta^n \otimes \alpha) \subset I(M^{n+k})$. Thus we have

COROLLARY. *If* $n \geqq k+2$, image $\tau_{n,k} \subset \theta^{n+k}(k)$.

De Sapio [4] has shown $\tau_{n,k}$ is trivial for $k \geqq n-3$. We shall, therefore, always take $n \geqq k+4$.

2. **Proof of the theorem.** Let $k \geqq 2$. Let $\Sigma^n \in I^n(k)$ and assume $d: M\#\Sigma \to M$ is a diffeomorphism for some k-connected closed M^n. Let N be an elementary cobordism between the disjoint union $M \cup \Sigma$ and $M\#\Sigma$; N can be taken to be the connected sum along the boundaries $M \times \{1\}$ and $\Sigma \times \{1\}$ of $M \times I$ and $\Sigma \times I$. Since N has $M \cup \Sigma \cup_h D^1$ as a deformation retract, it is also k-connected. Choose p in M so that p has a small n-disc neighborhood U in M with $U \times I$ disjoint from the $(n+1)$-disc removed from $M \times I$ when forming N. We can assume $d | U \times \{1\}$ is the identity map $d(u, 1) = u$. Identify $x \in M\#\Sigma$ with $(d(x), 0) \in M \times \{0\} \subset N$ and let the resulting manifold be V^{n+1}. Then $bV^{n+1} = \Sigma^n$. $p \times I \subset N$ becomes an embedded circle C in V with trivial normal bundle $C \times U$. It is elementary that $\pi_1(V)$ is generated by $j_*\pi_1(S^1)$ where $j: S^1 \to C$ is the embedding. Regarding V as $N \cup M \times I$, the Mayer-Vietoris sequence gives

$$0 \to H_1(V) \to Z + Z \xrightarrow{\lambda} Z + Z \xrightarrow{\mu} Z \to 0$$

where $\lambda(m, n) = (m+n, -m-n)$. Thus $H_1(V) \approx \pi_1(V) \approx Z$. The class $[j]$ is a generator. Moreover, the same Mayer-Vietoris sequence shows $H_i(V) = 0$, $2 \leqq i \leqq k$. Now, perform surgery on V by cutting out int $(C \times U)$ and attaching $D^2 \times S^{n-1}$ along the new boundary to obtain W^{n+1} which is also bounded by Σ^n. W is clearly simply connected and it is a simple matter to check, using $X = V \times I \cup_\phi D^2 \times D^n$, the trace of the surgery, that W is, in fact, k-connected.

3. **Remarks.** We consider a few special cases and mention several known results concerning the groups $I^n(k)$ and $\theta^n(k)$.

In a strict sense, the inclusion of the theorem is proper. $I^{2n-1}(n-1)$ is always trivial since an $(n-1)$-connected closed M^{2n-1} is a sphere Σ^{2n-1} and therefore has trivial inertia group. On the other hand, $\theta^{2n-1}(n-1)$ contains $\theta^{2n-1}(\partial\pi)$ which is, in general, nontrivial and, for even n, quite large.

Wall [13] has shown $I^{2n}(n-1)=0$ for $n\equiv 2, 3, 5, 6, 7 \pmod 8$ and has order no greater than 4, 8 or 2 respectively when $n\equiv 0, 1, 4 \pmod 8$ respectively. Also, $\theta^{2n}(n-1)=\theta^{2n}(\partial\pi)=0$ for $n\equiv 5, 6 \pmod 8$. If $n\geqq 4$ and $n\equiv 0, 1, 4 \pmod 8$ then $I^{2n}(n-1)=\theta^{2n}(n-1)$. For $n>8$, we choose a class $\alpha\in\pi_{n-1}(SO(n-1))$ such that $SS\alpha \in\pi_{n-1}(SO(n+1))$ is a generator and let M^{2n} be the total space of the S^n-bundle over S^n with characteristic class $S\alpha$. Then $I(M^{2n})=\theta^{2n}(n-1)$ [13, Theorem 10]. If $n=4$ or 8 take for M^{2n} Tamura's manifolds [12] $\overline{B}^8_{7,1}\cup_i D^8$ or $\overline{B}^{16}_{127,1}\cup_i D^{16}$; again, $I(M^{2n})=\theta^{2n}(n-1)$.

We now consider several low-dimensional cases. It is well known [5] that $\theta^7=\theta^7(3)=\theta^7(\partial\pi)\approx Z_{28}$. Tamura [11] has shown $I^7(2)=\theta^7$. Now, $\theta^8\approx Z_2$. The sole exotic 8-sphere is a spin-boundary [1] so $\theta^8=\theta^8(3)$. As remarked above, $I^8(3)=\theta^8(3)$. θ^8 has order eight. It follows from [1], [3], [8] that $\theta^9\approx Z_2+Z_2+Z_2$. Take as generators of the summands $\Sigma_1, \Sigma_2, \Sigma_3$. Then Σ_1 is not a spin-boundary while Σ_2 and Σ_3 are, so $\theta^9(3)\approx Z_2+Z_2$. Σ_3 may be taken to be the generator of $\theta^9(\partial\pi)\approx Z_2$. From [2], $\Sigma_3\in I^9(3)$. I do not know if $\Sigma_2\in I^9(3)$. $\theta^{10}\approx Z_2+Z_3$ with generators Σ_2 and Σ_3. Σ_2 is not a spin-boundary [8] while Σ_3 is, so $\theta^{10}(3)\approx Z_3$. $\theta^{10}(4)=\theta^{10}(\partial\pi)=0$. From the corollary, image $\tau_{7,3}\subset\theta^{10}(3)$, so $\tau_{7,3}: Z_{28}\to Z_3$ is trivial.

The author has learned that A. Winkelnkemper has shown that any exotic sphere $\Sigma^n\in\theta^n$ is in the inertia group of some closed M^n, but the author does not know whether Winkelnkemper's proof yields any information on the connectivity of M.

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