COMPLEX BORDISM OF CLASSIFYING SPACES

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ABSTRACT. It is shown that, for a finite group G , the Thom homomorphism maps the complex bordism $\Omega_*^U(B_G)$ onto the homology $H_*(B_G; \mathbb{Z})$ of a classifying space for G if and only if G has periodic cohomology.

Let B_G denote a classifying space for the finite group G , and let

$$\mu: \Omega_*^U(B_G) \rightarrow H_*(B_G; \mathbb{Z})$$

denote the Thom homomorphism ([5], [7]) from the complex bordism of B_G to its integral homology. It is well known (e.g. see [9]) that μ is onto if G is cyclic, and more generally if G has periodic cohomology [4, Chapter XII, §11], since then $H^n(B_G; \mathbb{Z}) = 0$ for all odd n , thus $H_p(B_G, \mathbb{Z}) = 0$ for all even $p > 0$ and this makes the complex bordism spectral sequence collapse. The main result of this note is the converse statement.

THEOREM 1. *If G is a finite group for which $\mu: \Omega_*^U(B_G) \rightarrow H_*(B_G; \mathbb{Z})$ is onto, then G has periodic cohomology.*

The main step in the proof of this result has recently been supplied by R. G. Swan [11].

THEOREM 2. *A finite group G with $H^n(B_G; \mathbb{Z}) = 0$ for all odd n must have periodic cohomology.*

We state the next result in order to give a more complete picture. For a CW-complex X write $\text{hom dim}_{\Omega_*^U} \Omega_*^U(X)$ for the projective dimension of the Ω_*^U -module $\Omega_*^U(X)$ [7].

THEOREM 3. *The following conditions are equivalent for a finite group G :*

- (a) *G has periodic cohomology, i.e. every abelian subgroup of G is cyclic [4, Chapter XII, §11].*
- (b) *$H^n(B_G; \mathbb{Z}) = 0$ for all odd n .*
- (c) *The Atiyah spectral sequence [2] $H^*(B_G; \mathbb{Z}) \Rightarrow K^*(B_G) = (R(G))^{\wedge}$ collapses.*

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- (d) *The Thom homomorphism $\mu: \Omega_*^U(B_G) \rightarrow H_*(B_G; \mathbf{Z})$ is onto.*
 (e) $\text{hom dim}_{\mathbb{Q}} \Omega_*^U(B_G) \leq 1$.

The implication (a) $\rightarrow$ (b) follows by reduction to the Sylow subgroups and a computation [4, Chapter XII]. The implications (b) $\rightarrow$ (c) and (b) $\rightarrow$ (d) are well-known consequences of the vanishing of the even differentials in the K -theory and complex bordism spectral sequences. Furthermore the implication (c) $\rightarrow$ (b) follows from the facts [2] that $K^1(B_G) = 0$ and that the K -theory filtration is even. The equivalence of (d) and (e) follows from Proposition 4 below.

We turn to the proof of Theorem 1. If the Thom homomorphism maps $\Omega_*^U(B_G)$ onto $H_*(B_G; \mathbf{Z})$, then the complex bordism spectral sequence

$$(1) \quad H_*(B_G; \Omega_*^U) \Rightarrow \Omega_*^U(B_G)$$

collapses. Let $K_*(\)$ denote the $\mathbf{Z}_2$ -graded homology theory dual to K -theory. J. W. Vick [12] has proved that $K_1(B_G)$ is isomorphic to the character group of the compact group $\tilde{K}^0(B_G)$, and that $\tilde{K}_0(B_G) = 0$. There is the K_* -spectral sequence

$$(2) \quad H_*(B_G; \mathbf{Z}) \Rightarrow K_*(B_G),$$

and we claim that it collapses if (1) does.

The natural transformation $\mu_c: \Omega_*^U(\) \rightarrow K_*(\)$ dual to the corresponding transformation from complex cobordism to K -theory [6] induces a morphism of the spectral sequence (1) into the spectral sequence (2). Since on the coefficient rings μ_c is the Todd genus and so maps Ω_*^U onto $K_*(pt) = \mathbf{Z}$, (2) indeed collapses if (1) does.

In the spectral sequence (2) we have $E_p^2 = H_p(B_G; \mathbf{Z})$ and

$$E_p^\infty \cong \text{Im}(K_p(B_G^p) \rightarrow K_p(B_G)) / \text{Im}(K_p(B_G^{p-1}) \rightarrow K_p(B_G)),$$

where B_G^k is the k -skeleton of the complex B_G , and thus $E_p^\infty = 0$ for even $p > 0$. If (2) collapses then $E^2 = E^\infty$ and so $H_p(B_G; \mathbf{Z}) = 0$ for all even $p > 0$. By the universal coefficient theorem it follows that $H^n(B_G; \mathbf{Z}) = 0$ for all odd n , and then Theorem 1 is a direct consequence of Theorem 2.

REMARK 1. A rapid but indirect proof that a finite p -group G with $H^n(B_G; \mathbf{Z}) = 0$ for all odd n must have periodic cohomology can be based on Lemma (2.1) and Proposition (6.1) of [10]. Then G is either a cyclic or generalized quaternion group.

REMARK 2. The same argument shows that the Thom homomorphism $\mu: \Omega_U^*(B_G) \rightarrow H^*(B_G; \mathbf{Z})$ from complex cobordism to cohomology

is onto if and only if G has periodic cohomology. In fact, if X is any complex and if either of the spectral sequences

$$H^*(X; \mathbb{Z}) \Rightarrow K^*(X), \quad H^*(X; \Omega_U^*) \Rightarrow \Omega_U^*(X)$$

collapses, then so does the other. This observation is due to T. tom Dieck [8, §6], and requires the theorem of Hattori and Stong; the homology analogue is also true.

The equivalence of (d) and (e) in Theorem 3 is a consequence of the next result.

PROPOSITION 4. *For any CW-complex X , $\mu: \Omega_*^U(X) \rightarrow H_*(X; \mathbb{Z})$ is onto if and only if $\text{hom dim}_{\Omega_*^U} \Omega_*^U(X) \leq 1$.*

For finite complexes the proof is given in [7, §3]. The following argument was suggested by Professor Larry Smith for the general case. We may regard X as a CW-spectrum in J. M. Boardman's stable category [3], and then X is (-1) -connected.

LEMMA 5. *Let X be an n -connected CW-spectrum. Then there exists a cofibration $A_1 \rightarrow A_0 \rightarrow X$ of n -connected CW-spectra such that $\Omega_*^U(A_0)$ is a free Ω_*^U -module and the sequence of induced homomorphisms*

$$0 \rightarrow \Omega_*^U(A_1) \rightarrow \Omega_*^U(A_0) \rightarrow \Omega_*^U(X) \rightarrow 0$$

is exact.

With the lemma in hand, it is entirely straightforward to show that the arguments of [7, §3] for finite complexes carry over to highly-connected CW-spectra and thus apply to all CW-complexes.

It remains to prove the lemma, and there is no loss of generality in supposing that X is (-1) -connected. Then the proof can be easily extracted from Lemmas 18 and 19 and Example (iv) on p. 29 of [1, Lecture 1]. It is only necessary to insist in Lemma 18 that the finite subcomplex X' have dimension $\leq p$, and to then choose E_α to be the p -skeleton of $E = MU$; then $S^p \wedge DE_\alpha$ has the homotopy type of a finite complex of dimension $\leq p$, and so is (-1) -connected. Now Lemma 19 yields the desired cofibration.

REMARK 3. For a finite group G , let $\text{rank}(G)$ denote the largest integer k such that G contains an abelian subgroup with minimum number of generators k . I conjecture that

$$(3) \quad \text{hom dim}_{\Omega_*^U} \Omega_*^U(B_G) = \text{rank}(G).$$

For example, we know that $\text{rank}(G)=1$ if and only if hom

$\dim_{\Omega_*^U} \Omega_*^U(B_G) = 1$. (If $\Omega_*^U(B_G)$ is a projective module then $H_*(B_G; \mathbf{Z})$ is free abelian, hence $\tilde{H}_*(B_G; \mathbf{Z}) = 0$ and so G is trivial.) If G is elementary abelian the results of [7, §5] show that

$$\text{hom dim}_{\Omega_*^U} \Omega_*^U(B_G) \geq \text{rank}(G),$$

but even in this case it is not generally known if $\Omega_*^U(B_G)$ has finite projective dimension.

Our final result concerns the relation between $\Omega_*^U(B_G)$ and $\Omega_*^U(B_H)$ when H is a Sylow p -subgroup of G . In this situation there is the following diagram of induced homomorphisms and transfer homomorphisms [5, §20]:

$$\begin{array}{ccccc} \Omega_n^U(B_G) & \xrightarrow{t} & \Omega_n^U(B_H) & \xrightarrow{i} & \Omega_n^U(B_G) \\ \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\ H_n(B_G; \mathbf{Z}) & \xrightarrow{t} & H_n(B_H; \mathbf{Z}) & \xrightarrow{i} & H_n(B_G; \mathbf{Z}). \end{array}$$

Let $\Omega_n^U(B_G)_{(p)}$ and $H_n(B_G; \mathbf{Z})_{(p)}$ denote the p -primary parts. It is well known that on $H_n(B_G; \mathbf{Z})$ the composition $i \circ t$ is multiplication by the index of H in G . Since this index is prime to p , it follows that $\tilde{H}_n(B_H; \mathbf{Z}) \xrightarrow{t} H_n(B_G; \mathbf{Z})_{(p)}$ is a split epimorphism.

PROPOSITION 6. *Let H be a Sylow p -subgroup of the finite group G . Then $i: \tilde{\Omega}_*^U(B_H) \rightarrow \Omega_*^U(B_G)_{(p)}$ is a split epimorphism of Ω_*^U -modules.*

PROOF. It suffices to show that $i \circ t$ is an automorphism of $\Omega_*^U(B_G)_{(p)}$. Both i and t induce morphisms of spectral sequences, yielding the following composition on the E^2 -terms:

$$H_*(B_G; \Omega_*^U) \xrightarrow{t} H_*(B_H; \Omega_*^U) \xrightarrow{i} H_*(B_G; \Omega_*^U).$$

Since this restricts to an isomorphism on the p -primary parts, it follows that $i \circ t$ yields an isomorphism on the p -primary parts of the E^∞ -terms. Since the complex bordism spectral sequence is convergent, it follows that $i \circ t$ is an automorphism of $\Omega_*^U(B_G)_{(p)}$.

REMARK 4. It follows that the projective dimension of the Ω_*^U -module $\Omega_*^U(B_G)$ is bounded by the maximum of the projective dimensions of the modules $\Omega_*^U(B_H)$ as H runs through the Sylow subgroups of G . Of course the rank of G is the maximum of the ranks of the Sylow subgroups of G . If the conjecture (3) holds for p -groups, and if

$$\text{hom dim}_{\Omega_*^U} \Omega_*^U(B_H) \leq \text{hom dim}_{\Omega_*^U} \Omega_*^U(B_G)$$

whenever H is a subgroup of G , then the conjecture is true in general. I would very much like to know an upper bound to $\text{hom dim}_{\Omega_*} \Omega_*^U(B_G)$ for p -groups.

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