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FIELDS IN WHICH VARIETIES HAVE RATIONAL POINTS: A NOTE ON A PROBLEM OF AX

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If k is a field, let $G(k)$ denote the Galois group over k of the algebraic closure of k. In [1] Ax proved the following result:

Let k be a perfect field such that $G(k)$ is abelian, and every (absolutely irreducible) variety defined over k has a k-rational point. Then $G(k)$ is pro-cyclic.

He asked [1, Problem 1] if the assumption that $G(k)$ is abelian can be removed. In this note we construct an example to show that this assumption is necessary. Recall that if X is a variety defined over k, then $k(X)$ is a regular extension of k [2]. Further X contains a $k(X)$-rational point (in fact a $k(X)$-rational point which is generic over k).

[Added in proof. A negative answer to Probelm 1 of [1] has also been given by Moshe Jarden (*Rational points on algebraic varieties over large number fields*, Bull. Amer. Math. Soc. **75** (1969), 603–606).]

If E and F are regular extensions of k, then the free composite of E and F over k is again a regular extension of k. Let $\{E_\alpha\}$ be any set of regular extensions of k. By an easy transfinite induction we can construct the free composite of $\{E_\alpha\}$ over k, and it will again be a regular extension of k.

Now let $\{E_\alpha\}$ be a complete set of nonisomorphic finitely generated regular extensions of k, and let $E(k)$ denote their free composite. Then $E(k)$ is a regular extension of k and every variety X, defined over k, has an $E(k)$-rational point. Set $E^n(k)=E(E^{n-1}(k))$ and $E^\infty(k)=\bigcup_{n=1}^\infty E^n(k)$.

If X is a variety defined over $E^\infty(k)$, then X is actually defined over $E^n(k)$ for some n, and hence has an $E^{n+1}(k)$-rational point. Since k is algebraically closed in $E^\infty(k)$, the map $G(E^\infty(k))\to G(k)$ is surjective, so $G(k)$ nonabelian implies $G(E^\infty(k))$ nonabelian. Finally if k is of characteristic zero, then $E^\infty(k)$ is perfect. Hence if we take, say $k=Q$, we get the desired example.

Remark. In an oral communication, S. Garfunkel has pointed out to Ax that Problem 5 of [1] has an affirmative solution, i.e. that the theory of the rings Z/m is decidable. This follows from Corollaries

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1 and 2 to Theorem 17 of [1], combined with Theorem 7.9 from reference 10 of [1].

References

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2. S. Lang, *Introduction to algebraic geometry*, Interscience, New York, 1958. MR **20** #7021.

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