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AN EXACT SOLUTION OF THE NONLINEAR DIFFERENTIAL EQUATION $\ddot{y}+p(t)y=q_m(t)/y^{2m-1}$

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ABSTRACT. An exact solution of the nonlinear differential equation $\ddot{y}+p(t)y=q_m(t)/y^{2m-1}$ is found to be $y=[u^m+c(m-1)^{-1}W^{-2}v]^{1/m}$ if $q_m(t)=c(uv)^{m-2}$. u and v are independent solutions of $\ddot{y}+p(t)y=0$ and W is their Wronskian.

E. Pinney [1] has shown that the nonlinear differential equation

$$\ddot{y} + p(t)y = c/y^3 \tag{1}$$

has exact solutions of the form

$$y = [u^2 + cW^{-2}v^2]^{1/2} \tag{2}$$

when $y(t_0)=y_0\neq 0$ and $\dot{y}(t_0)=\dot{y}_0$, for c an arbitrary constant and $p(t)$ given. The functions u and v are independent solutions of the linear equations

$$\ddot{y} + p(t)y = 0 \tag{3}$$

for which $u(t_0)=y_0$, $\dot{u}(t_0)=\dot{y}_0$, $v(t_0)=0$, $\dot{v}(t_0)\neq 0$, where their Wronskian $W=u\dot{v}-v\dot{u}=\text{const}\neq 0$.

With m real and finite, and $m\neq 0$, 1, it is possible to show that

$$y = [u^m + c(m-1)^{-1}W^{-2}v^m]^{1/m} \tag{4}$$

is an exact solution of

$$\ddot{y} + p(t)y = q_m(t)/y^{2m-1} \tag{5}$$

provided that u and v remain independent solutions of (3) and subject to the same conditions above, except that v_0 need not be zero, and provided that

$$q_m(t) = c(uv)^{m-2}. \tag{6}$$

The proof is simple and will be omitted.

Although $q_m(t)$ clearly restricts the general class of nonlinear equations implied by (5), important physical problems occur with

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initial conditions such as to make the solution (4) physically interesting. Moreover, it is interesting to regard the use of uv in $q_m(t)$ as a method for generating nonlinear differential equations, for which an exact solution is (4). The arbitrary choice of $p(t)$ allows a wide range of possibilities. Taking $p(t) = \pm\omega^2 = \text{const}$ provides two immediate examples:

$$\ddot{y} + \omega^2 y = c_1(\sin \omega t \cos \omega t)^{m-2}/y^{2m-1}, \tag{7}$$

$$\ddot{y} - \omega^2 y = c_1/y^{2m-1}, \tag{8}$$

having solutions

$$y = \left[a^m \cos^m \omega t + c_m b^m \sin^m \omega t\right]^{1/m}, \tag{9}$$

$$y = \left[a^m e^{m\omega t} + c_m' b^m e^{-m\omega t}\right]^{1/m}, \tag{10}$$

respectively, where

$$c_m = c_1/\left[\omega^2(ab)^m(m-1)\right] \quad \text{and} \quad c_m' = c_1/\left[4\omega^2(ab)^m(m-1)\right].$$

Constants a and b are determined by the initial conditions, while c_1, equal to $c(ab)^{m-2}$, is essentially arbitrary.

References

1. E. Pinney, *The nonlinear differential equation* $y''+p(x)y+cy^{-3}=0$, Proc. Amer. Math. Soc. 1 (1950), 681. MR 12, 336.

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