PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, January 1971

Ω-EXPLOSIONS

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ABSTRACT. In the present paper it is shown that if a flow satisfies Smale's Axiom A' and there is a cycle on its nonwandering set Ω, then the flow is not Ω-stable. This is done by "blowing up" the nonwandering set with a small perturbation. It is possible, in this setting, to give a characterization of Ω-stable flows when the nonwandering set is the union of a finite number of critical elements.

1. We describe here a certain class of flows for which there is, under small perturbations, a "blowing up" of the nonwandering set. The precise results are as follows.

Let M be a closed C^∞ manifold and let $\chi(M)$ be the set of C^r flows or vector fields on M with the C^r topology, $r \geqq 1$. For $X \in \chi(M)$ we denote its nonwandering set by $\Omega = \Omega(X)$. We say that X, $Y \in \chi(M)$ are Ω-conjugate if there is a homeomorphism $h:\Omega(X) \to \Omega(Y)$ sending trajectories of X into those of Y. $X \in \chi(M)$ is Ω-stable if for any $\epsilon > 0$ there is a neighborhood $N(X)$ in $\chi(M)$ such that if $Y \in N(X)$ then X is Ω-conjugate to Y by a homeomorphism which is $\in C^0$ close to the identity map in $\Omega(X)$.

We recall that $X \in \chi(M)$ satisfies Smale's Axiom A' if

(i) $\Omega = \Omega(X)$ is the disjoint union of the set of critical points F and the closure Λ of its periodic orbits,

(ii) each element of F is hyperbolic and Λ is a hyperbolic set for X (or the flow X_t).

See [1] or [6] for more details.

In this case, by the Spectral Decomposition Theorem [1], [6], Ω can be written as the disjoint union $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_k$ of closed invariant sets Ω_i and $X | \Omega_i$ is topologically transitive. The Ω_i are called basic sets. For each Ω_i we can define its stable and unstable manifolds $W^s(\Omega_i)$, $W^u(\Omega_i)$ [1] and

$$M = \bigcup_i W^s(\Omega_i) = \bigcup_i W^u(\Omega_i).$$

DEFINITION (1.1). Let X satisfy Axiom A'. We say that there is an n-cycle ($n \geqq 2$) on Ω if there is a sequence of basic sets $\Omega_0, \Omega_1, \cdots, \Omega_n$

Received by the editors November 13, 1969.
AMS 1969 subject classifications. Primary 3465.
Key words and phrases. Nonwandering set Ω, Ω-conjugate, Ω-stable, Smale's Axiom A', cycle property of Ω, characterization of Ω-stability.

(re-indexing the basic sets if necessary) such that $\Omega_0=\Omega_n$, $\Omega_i\neq\Omega_j$ otherwise and

$$W^s(\Omega_i)\cap W^u(\Omega_{i+1})\neq\emptyset \quad \text{for } 0\leqq i\leqq n-1.$$

The main purpose of this paper is to show

THEOREM (1.2). *If $X\in\chi(M)$ satisfies Axiom A′ and there is a cycle on Ω then X is not Ω-stable.*

That is, in presence of Axiom A′ the no-cycle property on Ω is a necessary condition for Ω-stability.

We also give, in the above terms, a characterization of Ω-stability when Ω is the finite union of critical points and closed orbits.

Theorem (1.2) is an extension for flows of the correspondent result for diffeomorphisms in [3]. It should be pointed out that in the diffeomorphism case, with a small perturbation we can enlarge Ω by introducing new transversal homoclinic points. For flows we do not always achieve this. This will be made clear in the proof of (1.2). In any case, however, we can produce a much larger Ω. Hence the name "Ω-explosion."

PROOF OF (1.2). Suppose there is an n-cycle ($n\geqq 2$) on Ω formed by the basic sets $\Omega_0, \Omega_1, \cdots, \Omega_n=\Omega_0$. We will show that there exists Z arbitrarily close to X but not Ω-conjugate to X by a small homeomorphism.

First we show that if $n>2$, then there is Y near X such that $Y|\Omega=X|\Omega$ and for which there is an $n-1$ cycle on Ω.

Consider two cases. First, suppose that all basic sets in the cycle are critical points. Let $x\in W^s(\Omega_0)\cap W^u(\Omega_1)$ and $y\in W^s(\Omega_1)\cap W^u(\Omega_2)$. From the λ-lemma [4], if V is any small cell transversal to $W^s(\Omega_1)$ at y then $x\in \mathrm{Cl}\, \mathcal{O}_+(V)$, where $\mathcal{O}_+(V)=\bigcup_{t\geq 0} X_t(V)$. Thus, there is a small C^r perturbation ΔX of X with small support containing x and y such that $Y=X+\Delta X$ satisfies the required properties. That is, $Y|\Omega=X|\Omega$ and $\Omega_0, \Omega_2, \cdots, \Omega_n$ form a cycle on $\Omega(Y)$.

Suppose now that not all basic sets $\Omega_0, \Omega_1, \Omega_2, \cdots, \Omega_n=\Omega_0$ consist of critical points. We claim that we can find Y near X such that $Y|\Omega=X|\Omega$ and for which there is an $n-1$ cycle, say $\Omega_0, \Omega_2, \cdots, \Omega_n=\Omega_0$, not all of these Ω_i being critical points. For if any of the Ω_i, say Ω_1, is a critical point we proceed as before to create the cycle $\Omega_0, \Omega_2, \cdots, \Omega_n=\Omega_0$. Otherwise, for each $z\in\Omega_i$ we have

$$\dim W^s\mathcal{O}(z)+\dim W^u\mathcal{O}(z)=\dim M+1,$$

where $\mathcal{O}(z)=\bigcup_t X_t(z)$. From the fact that

$$W^s(\Omega_i)=\bigcup_{z\in\Omega_i} W^s\mathcal{O}(z),\quad W^u(\Omega_i)=\bigcup_{z\in\Omega_i} W^u\mathcal{O}(z)$$

(see [2]) and the cycle property we get that one of the indices say $i=1$ is such that

$$\dim W^s\mathcal{O}(z_1) + \dim W^u\mathcal{O}(z_2) \geqq \dim M + 1,$$

where $z_2 \in \Omega_2$ and $z_1 \in \Omega_1$. If $x \in W^s(\Omega_0) \cap W^u(\Omega_1)$ and $y \in W^s(\Omega_1) \cap W^u(\Omega_2)$ then $x \in W^s\mathcal{O}(z_0) \cap W^u\mathcal{O}(z_1^*)$ and $y \in W^s\mathcal{O}(z_1) \cap W^u\mathcal{O}(z_2)$ for some $z_1, z_1^* \in \Omega_1$, $z_0 \in \Omega_0$ and $z_2 \in \Omega_2$. Since the sum of its dimensions is bigger than dim M, we can make $W^s\mathcal{O}(z_1)$ and $W^u\mathcal{O}(z_2)$ intersect transversally at y. This can be done by a small perturbation of X with small support containing y. Now, we may assume $\gamma = \mathcal{O}(z_1)$ to be periodic and z_1^* to be near γ for $X|\Omega_1$ is topologically transitive and the periodic orbits are dense in Ω_1. By the λ-lemma Cl $W^u\mathcal{O}(z_2) \supset W^s\gamma$, so x is near $W^u\mathcal{O}(z_2)$ and by a small perturbation of X near x we get for the resulting flow $W^u\mathcal{O}(z_2) \cap W^s\mathcal{O}(z_0) \neq \emptyset$, as claimed.

Continuing this process we end up with a flow Z arbitrarily near X so that $Z|\Omega = X|\Omega$ and $\Omega(Z)$ has points far from Ω. In fact, in the case where all the Ω_i forming the cycle are critical points we get $W^s(\Omega_0) \cap W^u(\Omega_0) \neq \emptyset$ and by the λ-lemma these points of intersection are nonwandering. Otherwise, we get a closed orbit γ in some Ω_i such that $W^s\gamma$ and $W^u\gamma$ intersect transversally outside $\Omega = \Omega(X)$. So we create new transversal homoclinic points which again are nonwandering. In both situations Z and X are not Ω-conjugate by a C^0 small homeomorphism. This finishes the proof of the theorem.

Remark. When $\Omega = \Omega(X)$ is the finite union of critical points and closed orbits, we can get the above result even when we relax the condition that the conjugacy should be C^0 small in the definition of Ω-stability. The same is true for diffeomorphisms in the general case (see [3]). For Ω-conjugacy for diffeomorphisms implies a one-one correspondence between periodic points of the same period. And in this case, the above process of "blowing up" Ω yields new transversal homoclinic points and by [5] new periodic points.

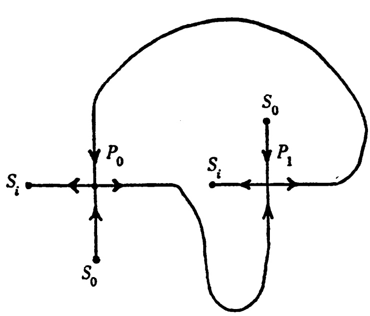

Figure 1

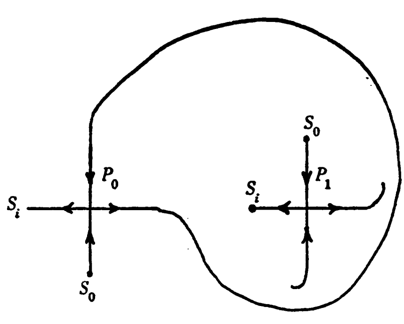

Figure 2

2. **Examples.** We show here some examples exhibiting the cycle property (see also [7]).

Consider in the sphere S^2 a flow X as in Figure 1, with $\Omega(X)$ finite and hyperbolic. The sources of X are denoted by S_0 and the sinks by S_i.

After a small perturbation we can get a vector field Z as in Figure 2, Notice that $W^s(\Omega_0)\cap W^u(\Omega_0)\subset\Omega(Z)$.

In the diffeomorphism case the range of perturbations is much larger. For instance, if we take the diffeomorphism induced at time one $f=X_{t=1}$, X as above, we get a diffeomorphism g near f exhibiting transversal homoclinic points as the point x in Figure 3.

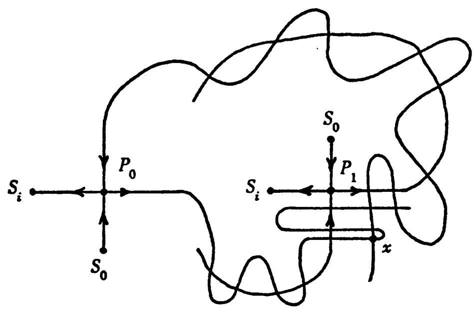

FIGURE 3

We can suspend these examples to S^3. By that we mean to extend the flow or diffeomorphism to S^3 leaving invariant S^2 as the equator and adding to Ω two sources: one at the north pole and one at the south pole. In this way we can build up in S^n examples of cycles on Ω. The same can be done in any manifold M. We take a Morse-Smale flow or diffeomorphism (see [4], [6]) with a point attractor (sink). Then, in a top dimensional disc around the attractor, we modify the flow or diffeomorphism to introduce one of the above examples.

3. **The problem of characterizing Ω-stability.** In this section we discuss this relevant problem and get in (3.4) a partial solution to the question.

First we state some results in this direction. Theorem (3.5) of [4] yields

THEOREM (3.1) [4]. *For $X\in\chi(M)$ if $\Omega=\Omega(X)$ is the finite union of hyperbolic critical points and closed orbits and has the no-cycle property then X is Ω-stable.*

A generalization of (3.1) is

THEOREM (3.2) [1]. *If $X \in \chi(M)$ satisfies Axiom* A′ *and $\Omega = \Omega(X)$ has the no-cycle property then X is Ω-stable.*

Theorem (3.2) is an extension for flows of Smale's Ω-stability theorem [7].

PROPOSITION (3.3). *Let $X \in \chi(M)$ be such that $\Omega = \Omega(X)$ is the finite union $\Omega = \bigcup_i \Omega_i$ of critical points and closed orbits. If X is Ω-stable then each Ω_i is hyperbolic.*

PROOF. Assume X to be Ω-stable and let $\Omega_j \subset \Omega$ be nonhyperbolic. The tangent bundle of M restricted to Ω_j, $T_{\Omega_j}M$, has a continuous splitting

$$T_{\Omega_j}M = E^s + E^u + E^c$$

where E^s, E^u, E^c are invariant by TX_t and, for $t>0$, TX_t contracts E^s and expands E^u, using some Riemannian metric on M. E^c corresponds to the "central" part of the splitting. Consider now a central manifold $W^c(\Omega_j)$ which is invariant by the flow, contains Ω_j and is tangent to E^c at Ω_j [1]. (Note that such a manifold is not unique in general.) Let U be a small neighborhood of Ω_j disjoint from all $\Omega_i \neq \Omega_j$. Leaving $W^c(\Omega_j)$ invariant, by a small perturbation of X with support in U we get a flow Y for which Ω_j is hyperbolic and $Y| W^c(\Omega_j)$ has Ω_j as a sink (attractor). U defines a neighborhood U_1 of Ω_j in $W^c(\Omega_j)$. From the fact that X is Ω-stable, there exists an open neighborhood V of Ω_j in $W^c(\Omega_j)$ so that $\overline{U}_1 \subset V$ and all points in V have, under Y, Ω_j as their Ω-limit set. That is, for all $x \in V$, we have $Y_t(x) \to \Omega_j$ as $t \to \infty$. Starting again with the flow X, we can get Z near X such that $\operatorname{supp}(Z-X) \subset U$, Z leaves $W^c(\Omega_j)$ invariant and $Z| W^c(\Omega_j)$ has Ω_j as a source. It is then clear that points in V will have, under Z, their ω-limit set in $U_1 - \Omega_j$. This creates new nonwandering points and so Z is not Ω-conjugate to X, contradicting the assumption. Thus the proposition is proved.

The above proof, due to C. Pugh and the author, works as well for the correspondent diffeomorphism case as stated in [3].

From (1.2), (3.1) and (3.3) we get

THEOREM (3.4). *Let X be such that $\Omega = \Omega(X)$ is the finite union $\Omega = \bigcup_i \Omega_i$ of critical points and closed orbits. Then X is Ω-stable if and only if each Ω_i is hyperbolic and Ω has the no-cycle property.*

For the general case, as in [8] we can pose the following

QUESTION. Does Ω-stability imply Axiom A′?

A positive answer to this question would lead, together with (3.2) and (1.2), to an important characterization of Ω-stability for flows: X is Ω-stable if and only if it satisfies Axiom A′ and $\Omega(X)$ has the no-cycle property.

References

1. M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, (to appear).

2. M. Hirsch, J. Palis, C. Pugh and M. Shub, *Neighborhoods of hyperbolic sets*, Invent. Math. **9** (1970), 121–134.

3. J. Palis, *A note on Ω-stability*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970.

4. ———, *On Morse-Smale dynamical systems*, Topology **8** (1969), 385–405.

5. S. Smale, *Diffeomorphisms with many periodic points*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press, Princeton, N. J., 1965, pp. 63–80. MR **31** #6244.

6. ———, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR **37** #3598.

7. ———, *The Ω-stability theorem*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970.

8. ———, *Global stability questions in dynamical systems*, Lectures in Modern Analysis and Applications, no. I, Springer-Verlag, New York, 1969, pp. 150–158.

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