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SETS OF LATTICE POINTS WHICH CONTAIN A MAXIMAL NUMBER OF EDGES[1]

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ABSTRACT. How should one select an l-element subset of a rectangular array of lattice points (points with integral coordinates) in n-dimensional Euclidean space so as to include the largest possible number of edges (pairs of points differing in exactly one coordinate)? It is shown that the generalized Macaulay theorem due to the author and B. Lindström contains the (known) solution.

1. Introduction and statement of results. Let $n\geqq 1$, $k_1\leqq k_2\leqq \cdots \leqq k_n$ and $l\leqq (k_1+1)(k_2+1)\cdots(k_n+1)=\theta$ be fixed positive integers. F_n denotes the θ n-tuples $\mathbf{x}=(x_1, x_2, \cdots, x_n)$ of integers x_i, $0\leqq x_i\leqq k_i$, $i=1, 2, \cdots, n$, ordered lexicographically—i.e. $\mathbf{x}<\mathbf{y}$ iff $x_i<y_i$ for the smallest integer i such that $x_i\neq y_i$. It will be helpful to imagine the elements of F_n arrayed in a matrix of k_1+1 rows and $\theta/(k_1+1)$ columns by writing them in increasing order from left to right and top to bottom.

Let A_l denote an l-element subset of F_n. An *edge* is an unordered pair $(\mathbf{x}, \mathbf{y})$ of n-tuples which disagree at exactly one place. A subset A of F_n *contains the edge* $(\mathbf{x}, \mathbf{y})$ if and only if $\mathbf{x}\in A$ and $\mathbf{y}\in A$. $E(A)$ denotes the number of edges A contains.

We now state two theorems. Theorem 1 is contained in Lindsey's paper [7] while Theorem 2 is Corollary 3 of the generalized Macaulay theorem [2]. The content of this paper is that these two theorems are equivalent.

THEOREM 1. *max* $E(A_l)=E(S_l)$ *where the maximum is taken over all l-element subsets of* F_n *and* S_l *denotes the first l elements of* F_n.

The sets A_l for which the maximum is attained are also characterized in Lindsey's paper.

In order to state the second theorem, we define the set-valued function Γ on F_n by $\Gamma(\varnothing)=\varnothing$,

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$$\Gamma(\mathbf{x}) = \{(x_1 - 1, x_2, \cdots, x_n), (x_1, x_2 - 1, x_3, \cdots, x_n), \cdots, (x_1, x_2, \cdots, x_n - 1)\} \cap F_n$$

and call a subset H of F_n *closed* if and only if $\Gamma(H) \subset H$, where $\Gamma(H) = \bigcup_{\mathbf{x} \in H} \Gamma(\mathbf{x})$. Notice that S_l is closed. Finally define

$$\alpha(\mathbf{x}) = x_1 + x_2 + \cdots + x_n \quad \text{and} \quad \alpha(H) = \sum_{\mathbf{x} \in H} \alpha(\mathbf{x}).$$

THEOREM 2. $\max \alpha(H_l) = \alpha(S_l)$ *where the maximum is taken over all closed l-element subsets of* F_n.

It is not difficult to verify that α and E agree on closed sets; hence $\alpha(S_l)$ can be replaced by $E(S_l)$ in the statement of Theorem 2. If this is done, the similarity between the two theorems becomes even greater. This similarity is noticed implicitly in the paper [**6**] of J. B. Kruskal. More precisely, Kruskal points out that the $k_1 = k_2 = \cdots = k_n = 1$ case of Theorem 1, which is contained in the papers of Harper [**3**] and Bernstein [**1**], is analogous to a result of his [**5**]. (Kruskal's result has been rediscovered by G. Katona and applied to a problem concerning the existence of certain subsets of a finite set [**4**].) Actually it can be shown that the $k_1 = k_2 = \cdots = k_n = 1$ case of Theorem 2 follows from Kruskal's result and that Kruskal's result contains the $k_1 = k_2 = \cdots = k_n = 1$ special case of the generalized Macaulay theorem.

2. **The equivalence of Theorems 1 and 2.** It is clear that Theorem 1 implies Theorem 2 since if one takes the maximum only over closed sets he has

$$\alpha(S_l) \leqq \max \alpha(H_l) = \max E(H_l) \leqq E(S_l) = \alpha(S_l).$$

Conversely, assume Theorem 2 and suppose that $\overline{A}_l$ is maximal: $E(\overline{A}_l) = \max E(A_l)$. We show that $\overline{A}_l$ can be replaced by S_l without decreasing the number of edges. This is obvious for 1-tuples. Assuming it is true for t-tuples, $t = 1, 2, \cdots, (n-1)$, we consider n-tuples. For a subset G of F_n, let G_i denote the elements of G which begin with i, $i = 0, 1, \cdots, k_1$; thus the elements of G_i appear in the ith row of F_n. Let a_i denote the number of elements in $(\overline{A}_l)_i$, $i = 0, 1, \cdots, k_1$. One easily convinces himself that it is no loss of generality to assume that $a_0 \geqq a_1 \geqq \cdots \geqq a_{k_1}$ since $\overline{A}_l$ could be replaced by a set having $E(\overline{A}_l)$ edges for which this is true. We will say that an edge $(\mathbf{x}, \mathbf{y})$ in $\overline{A}_l$ is an (i, j) edge $(i \leqq j)$, if $\mathbf{x} \in (\overline{A}_l)_i$ and $\mathbf{y} \in (\overline{A}_l)_j$. If $N_{(i,j)}(\overline{A}_l)$ is the number of (i, j) edges in $(\overline{A}_l)$, then

$$E(\bar{A}_l) = \sum_{r=0}^{k_1} \sum_{i=0}^{k_1-r} N_{(i,k_1-r)}(\bar{A}_l).$$

If $\bar{A}_l$ is replaced by the set $\bar{A}_l'$ consisting of the first a_i elements of $(F_n)_i$, $i=0, 1, \cdots, k_1$, no summand is decreased. That $N_{(i,i)}(\bar{A}_l') \geqq N_{(i,i)}(\bar{A}_l)$ follows from the $k_2, k_3, \cdots, k_n$ case of the induction hypothesis; that $N_{(i,j)}(\bar{A}_l') \geqq N_{(i,j)}(\bar{A}_l)$ if $i<j$ follows from the fact that $N_{(i,j)}(\bar{A}_l) \leqq a_j$ (since $(\bar{A}_l)_j$ has a_j elements) while $N_{(i,j)}(\bar{A}_l') = a_j$ (since the pth elements in $(\bar{A}_l')_i$ and $(\bar{A}_l')_j$ constitute an edge, $p=1, 2, \cdots, a_j$; we are using here that $a_i \geqq a_j$). Also $\bar{A}_l'$ is closed since if $\mathbf{x}$ is the pth element in $(\bar{A}_l')_i$, $1 \leqq p \leqq a_i$, then each element $\mathbf{z} \in \Gamma(\mathbf{x})$ is either a smaller element in $(F_n)_i$ and therefore in $(\bar{A}_l')_i$ since $(\bar{A}_l')_i$ is the *first* a_i elements of $(F_n)_i$, or $\mathbf{z}$ is the pth element of $(F_n)_{i-1}$ and therefore in $(\bar{A}_l')_{i-1}$ since $a_{i-1} \geqq a_i$.

Thus $\bar{A}_l$ has been replaced by a closed set $\bar{A}_l'$ having at least as many edges. If we now replace $\bar{A}_l'$ by S_l we again do not decrease the number of edges in view of Theorem 2 and the fact that α and E agree on closed sets. This completes the induction. The equality $\max E(\bar{A}_l) = E(S_l)$ now follows from

$$\alpha(S_l) = E(S_l) \leqq \max E(A_l) = E(\bar{A}_l) \leqq E(S_l) = \alpha(S_l).$$

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