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COMPACT TOTALLY $\mathcal{H}$ ORDERED SEMIGROUPS

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ABSTRACT. Compact totally $\mathcal{H}$ ordered semigroups are characterized. Each such semigroup is abelian and is, in fact, a closed subsemigroup of an I-semigroup. Several questions are posed about (algebraic) semigroups which are naturally totally (quasi-) ordered.

The structure of I-semigroups (or standard threads) has been known for some time [**3**], [**5**], [**10**]. One of the first steps in determining this structure was made by Faucett in showing that any I-semigroup is totally $\mathcal{H}$ ordered [**5**, Lemma 2]. In this paper we give a complete description of all compact totally $\mathcal{H}$ ordered semigroups. A consequence of the structure theorem is that each such semigroup is a closed subsemigroup of an I-semigroup and is hence abelian. The structure theorem for I-semigroups is a special case of our theorem. Several questions are posed at the end of the paper.

An I-semigroup is a topological semigroup on an arc in which one endpoint acts as an identity and the other acts as a zero. If S is a semigroup then S^1 is S if S has an identity and S^1 is S with an identity adjoined otherwise. Following [**7**], we define the following quasi-orders on a semigroup S:

$$
\begin{aligned}
x &\leqq (\mathcal{L})\ y && \text{if } S^1x \subset S^1y,\\
x &\leqq (\mathcal{R})\ y && \text{if } xS^1 \subset yS^1,\\
x &\leqq (\mathcal{H})\ y && \text{if } S^1x \subset S^1y \quad\text{and}\quad xS^1 \subset yS^1,\\
x &\leqq (\mathcal{D})\ y && \text{if } S^1xS^1 \subset S^1yS^1.
\end{aligned}
$$

If S is compact, each of these quasi-orders is a closed subspace of $S \times S$. If $\mathcal{K}$ denotes one of Green's equivalence relations [**7**], then it is clear that $\mathcal{K} = \leqq(\mathcal{K}) \cap [\leqq(\mathcal{K})]^{-1}$. We say that S is totally $\mathcal{K}$ quasi-ordered if each pair of elements of S compare relative to $\leqq(\mathcal{K})$. Also,

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S is totally $\mathcal{K}$ ordered if S is totally $\mathcal{K}$ quasi-ordered and $\leqq(\mathcal{K})$ is antisymmetric. If $\mathfrak{M}$ is a relation on S we say that $\mathfrak{M}$ is left compatible if $(zx, zy)\in\mathfrak{M}$ whenever $(x, y)\in\mathfrak{M}$ and $z\in S$. Right compatibility is defined dually and we say that $\mathfrak{M}$ is compatible if it is both left and right compatible. Of course, a transitive relation $\mathfrak{M}$ is compatible iff $(x, y)\in\mathfrak{M}$ and $(z, w)\in\mathfrak{M}$ imply $(xz, yw)\in\mathfrak{M}$. The minimal ideal of a semigroup S is denoted by $M(S)$. The closure of a set A is denoted by A^*. If $x\in S$ then $\theta(x)$ will denote the smallest subsemigroup of S containing x and $\Gamma(x)$ will denote $\theta(x)^*$. Recall that S is monothetic if $S=\Gamma(x)$ for some $x\in S$ [**6**], [**8**].

The main theorem (Theorem 1) will be preceded by seven lemmata, most of which are slight modifications of results already in the literature. The first lemma (and its dual) is an extension of A 3.19 and A 3.20 of [**7**] and is essentially due to Rothman [**10**]. Parts (i), (ii), (iv), and (v) are proved in [**7**] and (iii) is an immediate consequence of (ii).

LEMMA 1. *Let S be a compact semigroup which is totally $\mathcal{L}$ quasi-ordered. Then*

(i) $xS^1\subset S^1x$ *for all* $x\in S$.
(ii) $S^1x=S^1xS^1$ *for all* $x\in S$.
(iii) $\leqq(\mathcal{L})=\leqq(\mathcal{D})$.
(iv) $\mathcal{L}=\mathcal{D}$.
(v) $\mathcal{L}$ *is a congruence.*

LEMMA 2. *Let S be a compact totally $\mathcal{H}$ quasi-ordered semigroup. Then* $\leqq(\mathcal{H})=\leqq(\mathcal{L})=\leqq(\mathcal{R})=\leqq(\mathcal{D})$ *and hence* $\leqq(\mathcal{H})$ *is compatible,* $\mathcal{H}=\mathcal{L}=\mathcal{R}=\mathcal{D}$, $\mathcal{H}$ *is a congruence, and* S^1 *is normal.*

PROOF. This is an immediate consequence of Lemma 1.

The next lemma is analogous to Theorem 2 of [**4**].

LEMMA 3. *Let S be a compact totally $\mathcal{L}$ ordered semigroup with exactly one idempotent e. Then e acts as a zero for S.*

PROOF. Since $M(S)$ is a paragroup [**7**, A 1.23] and S contains exactly one idempotent, $M(S)$ is a group. However, as S is totally $\mathcal{L}$ ordered, each subgroup of S is trivial. It follows that $M(S)$ is a single point and that point is a zero for S.

The next lemma is analogous to Lemma 2.5 of [**4**].

LEMMA 4. *Let S be a compact totally $\mathcal{L}$ ordered semigroup with exactly one idempotent. Then S is a compact monothetic semigroup with zero.*

PROOF. According to Lemma 3, the unique idempotent of S is a zero for S. Let p be the largest element of S relative to the closed total order $\leqq(\mathcal{L})$. Let a be an arbitrary nonzero element of S. Since $0<(\mathcal{L})\ a\leqq(\mathcal{L})\ p$, $\{p^n\}$ converges to 0 ([6] or [8]), and $\{x\in S \mid x<(\mathcal{L})\ a\}$ is open containing 0, there exists a positive integer n such that $p^n<(\mathcal{L})\ a\leqq(\mathcal{L})\ p$. Let r be the unique positive integer such that $p^{r+1}<(\mathcal{L})\ a\leqq(\mathcal{L})\ p^r$. We show that $a=p^r$. Assume that $a\neq p^r$. Then $a=bp^r$ for some $b\in S$. Since $b\leqq(\mathcal{L})\ p$, $a=bp^r\leqq(\mathcal{L})\ p^{r+1}$. But $p^{r+1}<(\mathcal{L})\ a$, a contradiction. Hence $a=p^r$. Therefore, $S\backslash\{0\}\subset\theta(p)$. Now $0\in\Gamma(p)$ and so $\Gamma(p)=S$, completing the proof.

NOTE. It is well known that a compact monothetic semigroup with zero is either a finite cyclic semigroup with zero or is isomorphic to the subsemigroup $\Gamma(1/2)$ of the unit interval under ordinary multiplication (again [6] or [8]).

LEMMA 5. *Let S be a compact totally $\mathcal{H}$ ordered semigroup with exactly one idempotent. Then S is a compact monothetic semigroup with zero. In particular, S is abelian.*

PROOF. In view of Lemma 4, it suffices to show that S is totally $\mathcal{L}$ ordered. Lemma 2 yields that $\mathcal{H}=\mathcal{L}$; obviously S is totally $\mathcal{L}$ quasi-ordered, and it follows immediately that $\leqq(\mathcal{L})$ is a total order.

NOTATION. If S is a compact totally $\mathcal{H}$ ordered semigroup and x, $y\in S$ then $[x, y]$ will denote $\{z\in S \mid x\leqq(\mathcal{H})\ z\leqq(\mathcal{H})\ y\}$. Clearly $[x,y]$ is compact for each x, $y\in S$.

REMARK. For any semigroup S, $\mathcal{H}_{S^1}\cap(S\times S)=\mathcal{H}_S$. It follows that S is totally $\mathcal{H}$ ordered iff S^1 is totally $\mathcal{H}$ ordered.

The next lemma should be compared with Lemma 4 of [5] and the corollary on page 84 of [3].

LEMMA 6. *Let S be a compact totally $\mathcal{H}$ ordered semigroup with maximum element p. If e is an idempotent of S and $x\in S$ then $[0, x]$ is a subsemigroup of S, $[0, e]$ has identity e, and $[e, p]$ is a subsemigroup of S with zero e. If $a\in[0, e]$ and $b\in[e, p]$ then $ab=ba=a$.*

PROOF. Lemma 2 insures that $\leqq(\mathcal{H})$ is compatible and it follows immediately that $[0, x]$ and $[e, p]$ subsemigroups of S. That e acts as an identity for $[0, e]$ is a direct consequence of the definition of $\leqq(\mathcal{H})$. Now suppose that $e\leqq(\mathcal{H})\ y$. Since $e\leqq(\mathcal{H})\ e$ we have $e\leqq(\mathcal{H})\ ye$. Moreover, $y\leqq(\mathcal{H}_{S^1})\ 1$ implies $ye\leqq(\mathcal{H})\ e$ in view of the above remark. Hence $ye=e$ and dually $ey=e$. Suppose now that $a\in[0, e]$ and $b\in[e, p]$. Then

$$ab=(ae)b=a(eb)=ae=a=ea=(be)a=b(ea)=ba.$$

TERMINOLOGY. Any semigroup isomorphic to the unit interval with ordinary multiplication will be called a usual interval. Any semigroup isomorphic to the interval $[1/2, 1]$ with multiplication $x \circ y = \max \{1/2, xy\}$ will be called a nil interval.

LEMMA 7. *Let S be a compact totally $\mathcal{H}$ ordered semigroup. Let e and f be idempotents in S such that $e < (\mathcal{H}) f$ and $[e, f] \cap E(S) = \{e, f\}$. Then $[e, f]$ is a subsemigroup of S and is itself totally $\mathcal{H}$ ordered. Moreover, $[e, f]$ is either a usual interval, a nil interval, or a compact monothetic semigroup with zero e having isolated identity f adjoined.*

PROOF. That $[e, f]$ is a subsemigroup of S with zero e and identity f follows immediately from Lemma 6. It is not difficult to see that the $\mathcal{H}$ quasi-order on $[e, f]$ is the intersection of the $\mathcal{H}$ quasi-order on S with $[e, f] \times [e, f]$ and it follows that $[e, f]$ is totally $\mathcal{H}$ ordered.

Case 1. $\{f\}$ is not open in $[e, f]$. In this case there are many ways to see that $[e, f]$ is connected. For example, one could use the result of Mostert and Shields [**11**] to start a one parameter semigroup in $[e, f]$ at f and extend it to a connected subsemigroup containing e. Probably the easiest way to obtain a connected subset of $[e, f]$ containing e and f is to consider the closure of the union of all $\theta(x)$ with $e < (\mathcal{H})\, x < (\mathcal{H})\, f$ (see [**1**] and [**7**, Exercise 1, p. 133]). Once we obtain a connected subset of $[e, f]$ containing both e and f, a simple argument, using the fact that $[e, f]$ is totally ordered, yields that $[e, f] = C$. Hence, $[e, f]$ is an I-semigroup with exactly two idempotents. It now follows from [**3**], [**5**], and [**10**] that $[e, f]$ is either a usual or a nil interval.

Case 2. $\{f\}$ is open in $[e, f]$. In this case $[e, f)$ is a compact totally $\mathcal{H}$ ordered semigroup with exactly one idempotent e. Hence, by Lemma 5, $[e, f)$ is a compact monothetic semigroup with zero. Therefore, $[e, f]$ is a compact monothetic semigroup with zero e having isolated identity f adjoined.

Recall [**3**] that the contact extension of a semigroup S with identity 1 by a semigroup T with zero 0 is the semigroup $(S \cup T)/\rho$ where multiplication in $S \cup T$ is defined by

$$\begin{aligned} x \circ y &= xy && \text{if either } x, y \in S \text{ or } x, y \in T, \\ &= x && \text{if } x \in S \text{ and } y \in T, \\ &= y && \text{if } x \in T \text{ and } y \in S, \end{aligned}$$

and ρ is the congruence which identifies 0 with 1 in $S \cup T$.

THEOREM 1. *Let S be a compact totally $\mathcal{H}$ ordered semigroup with maximum element p and maximum idempotent e. Then $[0, e]$ is iso-*

morphic to a generalized hormos GHorm$(E(S), S_x, m_{xy})$ *where* S_x *is either a usual interval, a nil interval, or a compact monothetic semigroup with zero having isolated identity adjoined* [**7**]. *The entire semigroup* S *is isomorphic to the contact extension of* $[0, e]$ *by* $[e, p]$ *and* $[e, p]$ *is a compact monothetic semigroup with zero* e. *Moreover, each semigroup constructed in the above fashion is a compact totally* $\mathcal{H}$ *ordered semigroup.*

PROOF. That $[0, e]$ is isomorphic to GHorm$(E(S), S_x, m_{xy})$ follows from Lemmas 6 and 7 and from the proof of Theorem 3.1 of [**2**]. We point out that for each $f \in E(S)$, $S_f = \{f\}$ if f is not isolated from below in $E(S)$ and $S_f = [g, f]$ where $g = \sup([0, f) \cap E(S))$ otherwise. It follows from Lemmas 5 and 6 that $[e, p]$ is a compact monothetic semigroup with zero e. That S is isomorphic to the contact extension of $[0, e]$ by $[e, p]$ follows easily from Lemma 6. The converse is obvious.

One could also describe a compact totally $\mathcal{H}$ ordered semigroup (with identity) as an ordinal sum of a compact totally ordered set of half-open ligaments and one element semigroups where one allows as ligaments usual intervals, nil intervals, and compact monothetic semigroups with zero having isolated identity adjoined [**3**, p. 86–88].

COROLLARY 1. *Each compact totally* $\mathcal{H}$ *ordered semigroup is isomorphic to a closed subsemigroup of an I-semigroup.*

COROLLARY 2. *Each compact totally* $\mathcal{H}$ *ordered semigroup is abelian.*

Works related to the results of this paper may be found in [**9**] and [**13**]. Finally, we pose the following questions:

1. If S is a totally $\mathcal{H}$ quasi-ordered algebraic semigroup, must $\leqq(\mathcal{H})$ be compatible? Must $\mathcal{H}$ be a congruence? What if S is totally $\mathcal{H}$ ordered?

2. If S is a totally $\mathcal{H}$ ordered algebraic semigroup, must S be totally $\mathcal{L}$ ordered?

3. If S is a totally $\mathcal{L}$ ordered algebraic semigroup, must S be totally $\mathcal{H}$ ordered? We do not know the answer to this even if S is compact.

4. If S is a totally $\mathcal{H}$ ordered algebraic semigroup, must S be abelian?

REFERENCES

1. J. H. Carruth and J. D. Lawson, *On the existence of one-parameter semigroups*, Semigroup Forum 1 (1970), 85–90.

2. ———, *Semigroups through semilattices*, Trans. Amer. Math. Soc. 152 (1970), 597–608.

3. A. H. Clifford, *Connected ordered topological semigroups with idempotent endpoints*. I, Trans. Amer. Math. Soc **88** (1958), 80–98. MR **20** #1727.

4. ———, *Naturally totally ordered commutative semigroups*, Amer. J. Math. **76** (1954), 631–646. MR **15**, 930.

5. W. M. Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proc. Amer. Math. Soc. **6** (1955), 741–747. MR **17**, 173.

6. E. Hewitt, *Compact monothetic semigroups*, Duke Math. J. **23** (1956), 447–457. MR **17**, 1225.

7. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Charles E. Merrill, Columbus, Ohio, 1966.

8. R. J. Koch, *On monothetic semigroups*, Proc. Amer. Math. Soc. **8** (1957), 397–401. MR **19**, 290.

9. L. Megyesi and G. Pollák, *Über die Struktur der Hauptidealhalbgruppen*. I, Acta Sci. Math. (Szeged) **29** (1968), 261–270. MR **38** #3370.

10. P. S. Mostert and A. L. Shields, *On the structure of semi-groups on a compact manifold with boundary*, Ann. of Math. (2) **65** (1957), 117–143. MR **18**, 809.

11. ———, *One-parameter semigroups in a semigroup*, Trans. Amer. Math. Soc. **96** (1960), 510–517. MR **22** #6869.

12. N. J. Rothman, *Linearly quasi-ordered compact semigroups*, Proc. Amer. Math. Soc. **13** (1962), 352–357. MR **25** #1541.

13. A. R. Stralka, *Rees-simple semigroups*, J. London Math. Soc. **1** (1969), 705–708.

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