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L^2 ASYMPTOTES FOR THE KLEIN-GORDON EQUATION

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ABSTRACT. An approximation $a(x,t)$ is obtained for solutions $u(x,t)$ of the Klein-Gordon equation. $a(x,t)$ can be expressed in terms of the Fourier transforms of the Cauchy data and it is shown that $\|a(\cdot,t)-u(\cdot,t)\|_2\to 0$ as $t\to\infty$. This result is applied to show how energy distributes among various conical regions.

A wide class of solutions to the Klein-Gordon equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial^2 u}{\partial t^2} = u$$

can be written in the form

$$\begin{aligned} u(x, t) &= (2\pi)^{-n/2} \int e^{ix\cdot y}[F(y) \cos t\sqrt{(1 + y^2)} \\ &\qquad + G(y) \sin t\sqrt{(1 + y^2)}]d^n y \\ &= (2\pi)^{-n/2} \int e^{ix\cdot y}[\psi(y) \exp[it\sqrt{(1 + y^2)}] \\ &\qquad + \phi(y) \exp[-it\sqrt{(1 + y^2)}]]d^n y \end{aligned} \tag{1}$$

where $F=\psi+\phi$, $G=i(\psi-\phi)$ are in $L^2(R^n)$ and the integral over R^n is interpreted in the sense of Plancherel's theorem. Our main result is

THEOREM 1. *Define $a(x, t)=0$ for $|x|>t$ and for $|x|<t$ define*

$$a(x, t) = \left\{e^{i\theta(x,t)}\psi\left(\frac{-x}{\sqrt{(t^2 - x^2)}}\right) + e^{-i\theta(x,t)}\phi\left(\frac{x}{\sqrt{(t^2 - x^2)}}\right)\right\} \rho(x, t),$$

$$\theta(x, t) \equiv n\pi/4 + \sqrt{(t^2 - x^2)}, \qquad \rho(x, t) \equiv t(t^2 - x^2)^{-(n+2)/4}$$

where ψ and ϕ are the same L^2 functions as in (1). *Then*

$$\|u(\cdot, t) - a(\cdot, t)\|_2^2 \equiv \int | u(x, t) - a(x, t) |^2 d^n x \to 0$$

as $t\to\infty$.

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Before starting on the proof of Theorem 1 we mention a corollary. Define

$$\chi(x,t) = 1 \quad \text{if } |x| < t, \qquad = 0 \quad \text{otherwise.}$$

COROLLARY 1.

$$\lim_{t\to\infty} \|\chi u(\cdot,t)\|_2^2 = \|\psi\|_2^2 + \|\phi\|_2^2 = (\|F\|_2^2 + \|G\|_2^2)/2.$$

PROOF. By Theorem 1, $\lim_{t\to\infty}\|(1-\chi)u(\cdot,t)\|_2 = 0$ and hence

$$\lim_{t\to\infty} \|u(\cdot,t)\|_2^2 - \|\chi u(\cdot,t)\|_2^2 = 0.$$

Thus the corollary follows from the fact (see Brodsky [1]) that

$$\lim_{t\to\infty} \|u(\cdot,t)\|_2^2 = (\|F\|_2^2 + \|G\|_2^2)/2.$$

REMARK. Let V_n denote the volume of the unit ball in R^n so that $\|\chi u(\cdot,t)\|_2^2 \leq V_n t^n \|\chi u(\cdot,t)\|_\infty^2$. Applying the corollary one sees that if $\|\chi u(\cdot,t)\|_\infty = o(t^{-n/2})$ as $t\to\infty$ then $u=0$, a special case of a result by Littman [3].

The above corollary can be extended to the case where χ is replaced by the characteristic function of other cones (see Corollary 1′). Theorem 1 will be deduced from Theorem 1′ below. The proof of Theorem 1′ is based on

LEMMA 1. *Define*

$$W_t(x) = (2\pi)^{-n/2} \int e^{ix\cdot y} \exp[-it\sqrt{(1+y^2)}](1+y^2)^{-(n+2)/4} d^n y.$$

Then, for every $f \in L^1(R^n)$ and $\lambda \in R^n$,

$$\lim_{t\to\infty} t^{n/2} e^{in\pi/4} \exp[it/\sqrt{(1+\lambda^2)}] W_t * f\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right) = \hat{f}(\lambda).$$

PROOF. In [5] it is shown that for $t>0$,

$$W_t(x) = t^{-n/2} e^{-in\pi/4} \exp[-\sqrt{(x^2-t^2)}] + R_t(x) \tag{2}$$

where $\|R^t\|_\infty = O(t^{-1-n/2})$ as $t\to\infty$. By $\sqrt{(x^2-t^2)}$ we mean the value that lies on the positive imaginary axis when $|x|<t$ and on the positive real axis when $|x|>t$. Since $\|R_t\|_\infty = O(t^{-1-n/2})$ it is clear that $t^{n/2}\|R_t * f\|_\infty = O(t^{-1})$ and hence

$$\lim_{t\to\infty} t^{n/2} e^{in\pi/4} \exp[it\sqrt{(1+\lambda^2)}] R_t * f\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right) = 0.$$

Thus to complete the proof of the lemma we must show

$$\lim_{t\to\infty} (2\pi)^{-n/2} \int \exp\left(\frac{it}{\sqrt{(1+\lambda^2)}} - \sqrt{\left(\left|\frac{\lambda t}{\sqrt{(1+\lambda^2)}} - x\right|^2 - t^2\right)}\right) f(x)\, d^n x$$

$$= (2\pi)^{-n/2} \int e^{-i\lambda\cdot x} f(x) d^n x \equiv \hat{f}(\lambda).$$

But this is a consequence of Lebesgue's dominated convergence theorem because

$$\lim_{t\to\infty} \frac{it}{\sqrt{(1+\lambda^2)}} - \sqrt{\left(\left|\frac{\lambda t}{\sqrt{(1+\lambda^2)}} - x\right|^2 - t^2\right)} = -i\lambda\cdot x.$$

For $\phi \in L^2(R^n)$ define

$$U_t\phi(x) = (2\pi)^{-n/2} \int e^{ix\cdot y} \exp[-it\sqrt{(1+y^2)}]\phi(y) d^n y. \tag{3}$$

COROLLARY 2. *If there exists* $f \in L_1(R^n)$ *such that* $\hat{f}(\lambda) = (1+\lambda^2)^{(n+2)/4} \cdot \phi(y)$ *then*

(i) $\|U_t\phi\|_\infty \leqq (2\pi)^{-n/2}\|f\|_1\|W_t\|_\infty \sim (2\pi)^{-n/2}\|f\|_1 t^{-n/2}$ *as* $t \to \infty$,

(ii) $\displaystyle\lim_{t\to\infty} t^{n/2} e^{i\alpha(\lambda,t)} U_t\phi\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right) = \hat{f}(\lambda) = (1+\lambda^2)^{(n+2)/4}\phi(\lambda)$,

where $\alpha(\lambda, t) \equiv n\pi/4 + t/\sqrt{(1+\lambda^2)}$.

PROOF. Both (i) and (ii) follow easily from the fact that

$$U_t\phi(x) = (2\pi)^{-n/2} \int e^{ix\cdot\lambda} \exp[-it\sqrt{(1+\lambda^2)}](1+\lambda^2)^{-(n+2)/4}\hat{f}(\lambda) d^n\lambda$$

$$= (2\pi)^{-n/2} \int W_t(x-y) f(y) d^n y \equiv W_t * f(x).$$

REMARK. The proof of Corollary 2 follows the approach used by Brodsky [2] and Segal [7, pp. 95–98] to obtain bounds like that given by (i). Recently, I became aware of a different approach by Littman [4] which when applied to the present situation yields (i) and (ii) with different assumptions on ϕ. Using Littman's approach Theorem 1 can be extended to more general situations (see [6]).

Motivated by (ii) of Corollary 2 we define an approximation $A_t\phi(x)$ to $U_t\phi(x)$ by requiring

$$t^{n/2}e^{i\alpha(\lambda,t)}A_t\phi\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right) = (1+\lambda^2)^{(n+2)/4}\phi(\lambda), \quad \lambda \in R^n, \ t > 0. \tag{4}$$

To see how this works out consider the transformation

$$T_t\colon \lambda \to x = \frac{\lambda t}{\sqrt{(1+\lambda^2)}}$$

that maps R^n onto the ball $|x| < t$. Since $x^2 = \lambda^2 t^2/(1+\lambda^2)$ we have $\lambda^2(t^2-x^2) = x^2$ and hence $T_t^{-1}\colon x \to \lambda = x/\sqrt{(t^2-x^2)}$. Taking $\lambda = T_t^{-1}(x)$ in (4) and multiplying by $t^{-n/2}e^{-i\alpha}$ gives

$$\begin{aligned} A_t\phi(x) &= \exp[-i\alpha(T_t^{-1}(x), t)]t^{-n/2}\left\{1 + \frac{x^2}{t^2 - x^2}\right\}^{(n+2)/4} \phi(T_t^{-1}(x)) \\ &= e^{-i\theta(x,t)}\rho(x,t)\phi\left(\frac{x}{\sqrt{(t^2-x^2)}}\right), \qquad |x| < t, \end{aligned} \tag{5}$$

where θ and ρ are the functions defined in Theorem 1.

THEOREM 1′. *For $\phi \in L^2(R^n)$ define $U_t\phi$ and $A_t\phi$ by* (3) *and* (5). *Then*

(i) $\|A_t\phi\|_{2,t} = \|\phi\|_2 = \|U_t\phi\|_2$,
(ii) $\lim_{t\to\infty}\|U_t\phi - A_t\phi\|_{2,t} = 0$,
(iii) $\lim_{t\to\infty}\int_{|x|>t}|U_t\phi(x)|^2dx = 0$,

where

$$\|f\|_{2,t} \equiv \left\{\int_{|x|<t} |f(x)|^2 d^n x\right\}^{1/2}.$$

PROOF. It is not difficult to check that the Jacobian of T_t is

$$\frac{\partial(x_1, \cdots, x_n)}{\partial(\lambda_1, \cdots, \lambda_n)} = t^n(1+\lambda^2)^{-(n+2)/2} = \rho^{-2}(T_t(\lambda), t).$$

Thus by the change of variable theorem for multiple integrals

$$\|f\|_{2,t}^2 = \int |f(T_t(\lambda))|^2 \rho^{-2}(T_t(\lambda), t)d^n\lambda. \tag{6}$$

Taking $f = A_t\phi$ in (6) we obtain the left-hand side of (i). The other half of (i) is Parseval's equality.

To prove (ii) let $\epsilon > 0$ and choose $\tilde\phi \in C_c^\infty(R^n)$ such that $\|\phi - \tilde\phi\|_2 < \epsilon/3$. Applying (i) to $\phi - \tilde\phi$ we have

$$\|U_t\phi - U_t\tilde{\phi}\|_{2,t} \leqq \|U_t\phi - U_t\tilde{\phi}\|_2 = \|A_t\phi - A_t\tilde{\phi}\|_{2,t} = \|\phi - \tilde{\phi}\|_2.$$

Thus

$$\|U_t\phi - A_t\phi\|_{2,t} \leqq \|U_t\phi - U_t\tilde{\phi}\|_{2,t} + \|U_t\tilde{\phi} - A_t\tilde{\phi}\|_{2,t} + \|A_t\tilde{\phi} - A_t\phi\|_{2,t}$$
$$< \epsilon/3 + \|U_t\tilde{\phi} - A_t\tilde{\phi}\|_{2,t} + \epsilon/3,$$

so to prove (ii) we need only show there exists τ such that $t>\tau$ implies $\|U_t\tilde{\phi}-A_t\tilde{\phi}\|_{2,t}<\epsilon/3$. Since this amounts to proving (ii) with ϕ replaced by $\tilde{\phi}$, we simply assume $\phi\in C_c^\infty(R^n)$.

Applying (6) we have

$$\|U_t\phi - A_t\phi\|_{2,t}^2 = \int \left| U_t\phi\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right) - A_t\phi\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right)\right|^2 \cdot \rho^{-2}(T_t(\lambda), t)d^n\lambda \tag{7}$$
$$= \int |e^{-i\alpha(\lambda,t)}[g_t(\lambda) - \phi(\lambda)]|^2 d^n\lambda$$

where

$$g_t(\lambda) = e^{i\alpha(\lambda,t)}t^{n/2}(1+\lambda^2)^{(n+2)/4}U_t\phi\left(\frac{\lambda t}{\sqrt{(1+\lambda^2)}}\right).$$

Clearly $\|U_t\phi\|_{2,t}^2=\int|g_t(\lambda)|^2d^n\lambda$ and hence

$$\|g_t\|_2 \leqq \|U_t\phi\|_2 = \|\phi\|_2. \tag{8}$$

Since $\phi\in C_c^\infty(R^n)$ part (ii) of Corollary 2 can be used to conclude that for every $\lambda\in R^n$

$$\lim_{t\to\infty} g_t(\lambda) = \phi(\lambda). \tag{9}$$

An application of Fatou's lemma and Egoroff's theorem shows that (8) and (9) imply $\lim_{t\to\infty}\|g_t-\phi\|_2=0$ which in view of (7) establishes (ii).

To prove (iii) we must show $\lim_{t\to 0}\|U_t\phi\|_2^2-\|U_t\phi\|_{2,t}^2=0$ which is obvious from (i) and (ii).

Proof of Theorem 1. Since

$$u(x,t) = \overline{U_t\bar{\psi}(-x)} + U_t\phi(x), \qquad a(x,t) = \overline{A_t\bar{\psi}(-x)} + A_t\phi(x),$$

Theorem 1 follows immediately from Theorem 1′.

Let Γ be a cone inside the cone $\{(x,t):|x|<t\}$. That is, assume $(x,t)\in\Gamma$ implies $|x|<t$ and $(sx,st)\in\Gamma$ for all $s>0$. Put

$$B = \left\{\frac{x}{\sqrt{(1 - x^2)}} : (x, 1) \in \Gamma\right\}$$

and let $\beta(\lambda)$ be the characteristic function of B. Then the characteristic function $\gamma(x, t)$ of Γ is zero for $|x| > t$ and satisfies

$$\gamma(x, t) = \beta \circ T_t^{-1}(x) = \beta\left(\frac{x}{\sqrt{(t^2 - x^2)}}\right) \quad \text{for } |x| < t.$$

COROLLARY 1′. *Let $\gamma(x, t)$, B be as above and let $u(x, t)$, $\phi(y)$, $\psi(y)$ be as in Theorem 1. Then*

$$\lim_{t \to \infty} \|\gamma u(\cdot, t)\|_2^2 = \int_B |\psi(-\lambda)|^2 + |\phi(\lambda)|^2 d^n\lambda.$$

REMARK. Similar expressions can be obtained for the kinetic and potential energies $\|\gamma u_t\|_2^2$, $\|\gamma u\|_2^2 + \sum \|\gamma u_{x_i}\|_2^2$ since u_t and u_{x_i} can also be written in the form of equation (1). This gives an extension of the Virial theorem stated in Brodsky [1].

PROOF. By Theorem 1 it suffices to prove the corollary with $u(x, t)$ replaced by $a(x, t)$. When $|x| < t$ we have

$$\gamma(x, t)a(x, t) = \beta\left(\frac{x}{\sqrt{(t^2 - x^2)}}\right)\left\{e^{i\theta(x,t)}\psi\left(\frac{-x}{\sqrt{(t^2 - x^2)}}\right)\right.$$
$$\left. + e^{-i\theta(x,t)}\phi\left(\frac{x}{\sqrt{(t^2 - x^2)}}\right)\right\}\rho(x, t),$$

and hence, by (6),

$$\|\gamma a(\cdot, t)\|_{2,t}^2 = \int \beta(\lambda) \left| e^{i\alpha(\lambda,t)}\psi(-\lambda) + e^{-i\alpha(\lambda,t)}\phi(\lambda)\right|^2 d^n\lambda.$$

Since $|e^{i\alpha}\psi + e^{-i\alpha}\phi|^2 = |\psi|^2 + |\phi|^2 + e^{2i\alpha}\psi\bar{\phi} + e^{-2i\alpha}\bar{\psi}\phi$ the corollary follows from

LEMMA 2. *For $f \in L^1(R^n)$ define $I_f(t) = \int \exp[it/\sqrt{(1+\lambda^2)}]f(\lambda)\, d^n\lambda$. Then $\lim_{|t| \to \infty} I_f(t) = 0$.*

PROOF. Since $C_c(R^n)$ is dense in $L^1(R^n)$ and since $|I_f(t) - I_g(t)| \leq \|f - g\|_1$ it suffices to prove the lemma when f is continuous and has compact support. Switching to spherical coordinates gives

$$I_f(t) = \int_0^\infty \exp[it/\sqrt{(1 + r^2)}]F(r)dr, \qquad F(r) = r^{n-1}\int_{|y|=1} f(ry)dS(y).$$

Applying the change of variable $u = 1/\sqrt{(1 + r^2)}$ yields

$$I_f(t) = \int_0^1 e^{itu}\phi(u)du, \qquad \phi(u) = F\left(\frac{\sqrt{(1-u^2)}}{u}\right)\frac{1}{u^2\sqrt{(1-u^2)}}\cdot$$

Since $\sqrt{(1-u^2)}\ \phi(u) \in C_c((0,\ 1])$, the lemma follows from the Riemann-Lebesgue theorem.

REMARK ON L^p BEHAVIOR. By Hölder's inequality

$$\begin{aligned}\|\gamma u(\cdot,t)\|_2^2 &\leqq \|\gamma(\cdot,t)\|_q\|\gamma u^2(\cdot,t)\|_{q'}\\ &= \|\gamma(\cdot,t)\|_1^{1/q}\|\gamma u(\cdot,t)\|_{2q'}^2\\ &= (\|\gamma(\cdot,1)\|_1 t^n)^{(1-2/p)}\|\gamma u(\cdot,t)\|_p^2\end{aligned}$$

where $1-1/q=1/q'=2/p$. Thus Corollary 1′ gives a lower bound for $\liminf_{t\to\infty} t^{n/2}t^{-n/p}\|\gamma u(\cdot,t)\|_p$, $p\geqq 2$, which is positive unless $\phi(\lambda)$ and $\psi(-\lambda)$ vanish for $\lambda\in B$.

On the other hand, by using Corollary 2(i) and the estimate $\|f\|_p^p\leqq\|f\|_2^2\|f\|_\infty^{p-2}$, $2\leqq p<\infty$ one can show

$$\|u(\cdot,t)\|_p = O(t^{-n/2}t^{n/p}), \qquad \|u(\cdot,t)-a(\cdot,t)\|_p = o(t^{-n/2}t^{n/p})$$

as $t\to\infty$, provided ϕ and ψ satisfy the condition of Corollary 2.

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