DEVELOPABLE SPACES AND p-SPACES[1]

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ABSTRACT. In this paper it is shown that the following statements are equivalent, for a completely regular space X.
(i) X is a developable space.
(ii) X is a strict p-space with a $\overline{G}_\delta$-diagonal.
(iii) X is a θ-refinable p-space with a G_δ-diagonal.

1. **Introduction.** Developable spaces and p-spaces have been studied in connection with problems of metrizability. The concept of a developable space can be traced back to the Alexandroff-Urysohn metrization theorem [1]. Later it was shown by R. H. Bing [5] that every paracompact developable space is metrizable. Plumed spaces or p-spaces were first studied by A. V. Arhangel'skiĭ [2]. It has been shown by A. Okuyama [12] that a paracompact p-space with a G_δ-diagonal is metrizable.

Recently two important relationships between developable spaces and p-spaces have been established. Burke and Stoltenberg [8] proved that a completely regular space is developable iff it is a p-space with a σ-discrete network, and G. Creede [11] proved that a completely regular space is developable iff it is a semistratifiable p-space.

In light of the results stated above, it is natural to ask about the relationship between developable spaces and p-spaces having a G_δ-diagonal. We first state some important definitions and propositions.

All topological spaces are assumed to be Hausdorff. If Y is a space and $U \subset X \subset Y$, then $\mathrm{Cl}_X(U)$ and $\mathrm{Cl}_Y(U)$ will denote the closure of U in X and Y respectively. If γ and γ' are covers of X, then $\gamma \wedge \gamma' = \{G \cap H \mid G \in \gamma, H \in \gamma'\}$, and $\gamma' < \gamma$ means that γ' refines γ. The term "integer," when referring to an index, will always mean "positive integer".

A space X is *developable* iff there is a sequence $\{\gamma_n \mid n = 1, 2, \cdots\}$ of

open covers of X such that, for each $x \in X$, $\{\mathrm{St}(x, \gamma_n) \mid n = 1, 2, \cdots\}$ is a local base at x. Such a sequence of covers is called a *development* for X. It is well known that every metrizable space is developable, and every developable space is clearly first countable. Regular developable spaces are also known as Moore spaces.

A completely regular space X is a *p-space* iff, for some compactification bX of X, there is a sequence $\Gamma = \{\gamma_n \mid n = 1, 2, \cdots\}$ of covers of X, by sets open in bX, with the property that, for each $x \in X$, $\cap\{\mathrm{St}(x, \gamma_n) \mid n = 1, 2, \cdots\} \subset X$. The sequence of covers Γ is called a *feathering* of X in bX. Γ is called a *strict feathering* iff, for each $x \in X$ and each n, there is an integer m such that $\mathrm{Cl}_{bX}(\mathrm{St}(x, \gamma_m)) \subset \mathrm{St}(x, \gamma_n)$. X is a *strict p-space* iff X has a strict feathering in some compactification.

The definitions of p-space and strict p-space are due to A. V. Arhangel'skiĭ, and he has obtained many interesting results concerning them. (See [2], [3] and [4].) In particular, if X has a (strict) feathering in some compactification bX, then X has a (strict) feathering in its Stone-Čech compactification βX. All metrizable spaces are strict p-spaces, and all locally compact spaces are p-spaces.

A space X has a *G_δ-diagonal* iff $\Delta_X = \{(x, x) \mid x \in X\}$ is a G_δ-subset of $X \times X$. It is easily seen that this is equivalent to the condition:

(a) There is a sequence $\{\gamma_n \mid n = 1, 2, \cdots\}$ of open covers of X such that if $x, y \in X$ and $x \neq y$, then, for some n, $y \notin \mathrm{St}(x, \gamma_n)$.

If, in addition, the sequence $\{\gamma_n \mid n = 1, 2, \cdots\}$ satisfies

(b) for each $x \in X$ and each n, there is an integer m such that $\mathrm{Cl}(\mathrm{St}(x, \gamma_m)) \subset \mathrm{St}(x, \gamma_n)$, then X is said to have a $\overline{G_\delta}$-*diagonal*. This definition is due to C. R. Borges [6].

Just as it is possible to define uniform structures either in terms of open covers or in terms of relations (i.e. entourages), so it is possible to characterize each of the properties defined above in terms of relations. In general, given an open cover γ of a space X, the relation $R = \cup\{G \times G \mid G \in \gamma\}$ is an open neighborhood of Δ_X and, for each $x \in X$, $\mathrm{St}(x, \gamma) = R[x] = \{y \in X \mid (x, y) \in R\}$. On the other hand, if R is an open neighborhood of Δ_X, then $\gamma = \{G \subset X \mid G \text{ is open and } G \times G \subset R\}$ is an open cover of X such that, for each $x \in X$, $\mathrm{St}(x, \gamma) \subset R[x]$. If R is symmetric, then $\mathrm{St}(x, \gamma) = R[x]$. The following propositions are proved by straightforward applications of the principle just stated.

PROPOSITION 1.1. *A space X is developable iff there is a sequence $\{V_n \mid n = 1, 2, \cdots\}$ of open neighborhoods of Δ_X such that, for each $x \in X$, $\{V_n[x] \mid n = 1, 2, \cdots\}$ is a local base at x.*

PROPOSITION 1.2. *Let X be a completely regular space and bX any compactification of X. X has a feathering in bX iff there is a sequence $\{V_n \mid n = 1, 2, \cdots\}$ of open subsets of $bX \times bX$ satisfying*
   (i) *$\Delta_X \subset V_n$, for each n;*
   (ii) *for each $x \in X$, $\cap \{V_n[x] \mid n = 1, 2, \cdots\} \subset X$.*

PROPOSITION 1.3. *A space X has a strict feathering in bX, iff there is a sequence of symmetric neighborhoods $\{V_n \mid n = 1, 2, \cdots\}$, open in $bX \times bX$, satisfying* (i) *and* (ii) *of Proposition 1.2 as well as*
   (iii) *for each $x \in X$ and each n, there is an integer m such that $\mathrm{Cl}_{bX}(V_m[x]) \subset V_n[x]$.*

PROPOSITION 1.4. *A space X has a $\overline{G}_\delta$-diagonal iff there is a sequence $\{V_n \mid n = 1, 2, \cdots\}$ of symmetric open neighborhoods of Δ_X satisfying*
   (a) *$\cap \{V_n \mid n = 1, 2, \cdots\} = \Delta_X$;*
   (b) *for each $x \in X$ and each n, there is an integer m such that $\mathrm{Cl}(V_m[x]) \subset V_n[x]$.*

REMARK. In each of the preceding propositions, we may assume, without loss of generality, that the sequence of relations $\{V_n \mid n = 1, 2, \cdots\}$ has the property that $V_{n+1} \subset V_n$, for each n. For if $\{V_n \mid n = 1, 2, \cdots\}$ satisfies the conditions in any one of the Propositions 1.1–1.4, then so does the sequence $\{V_n' \mid n = 1, 2, \cdots\}$ where $V_n' = \cap \{V_i \mid 1 \leq i \leq n\}$.

2. **Strict p-spaces with $\overline{G}_\delta$-diagonals.** The following theorem is a corollary of two results of A. V. Arhangel'skiĭ (stated but not proved in [4]). We give a self-contained proof here.

THEOREM 2.1. *A completely regular developable space is a strict p-space.*

PROOF. Let X be a completely regular developable space, bX a compactification of X, and $\{\gamma_n \mid n = 1, 2, \cdots\}$ a development for X. For each n and each $G_n \in \gamma_n$, choose V_n open in bX such that $V_n \cap X = G_n$. For each n, denote the collection of all such sets V_n by μ_n. We will show that $\{\mu_n \mid n = 1, 2, \cdots\}$ is a strict feathering of X in bX.

Let $x \in X$ and let U be any neighborhood of x, open in bX. Since bX is regular, there is a neighborhood W of x, open in bX, such that $\overline{W}^{bX} \subset U$. Since $\{\gamma_n \mid n = 1, 2, \cdots\}$ is a development, there exists an integer m such that $\mathrm{St}(x, \gamma_m) \subset W \cap X \subset W$. Moreover, $\mathrm{St}(x, \gamma_m) = X \cap \mathrm{St}(x, \mu_m)$, so $\mathrm{Cl}_{bX}(\mathrm{St}(x, \gamma_m)) = \mathrm{Cl}_{bX}(\mathrm{St}(x, \mu_m))$, since X is dense in bX and $\mathrm{St}(x, \mu_m)$ is open in bX. Consequently,

$$\mathrm{St}(x, u_m) \subset \mathrm{Cl}_{bX}(\mathrm{St}(x, \mu_m)) = \mathrm{Cl}_{bX}(\mathrm{St}(x, \gamma_m)) \subset \overline{W}^{bX} \subset U.$$

It follows that $\{\mu_n \mid n = 1, 2, \cdots\}$ is a strict feathering of X in bX. In fact, $\cap \{\text{St}(x, \mu_n) \mid n = 1, 2, \cdots\} = \{x\}$, for each $x \in X$. Strictness can be seen by letting $U = \text{St}(x, \gamma_n)$. ■

PROPOSITION 2.2. *A regular developable space has a $\overline{G}_\delta$-diagonal.*

PROOF. Let X be a regular space with a development $\{\gamma_n \mid n = 1, 2, \cdots\}$. It is easily verified that this sequence of covers satisfies (a) and (b) in the definition of $\overline{G}_\delta$-diagonal. ■

THEOREM 2.3. *A strict p-space with a $\overline{G}_\delta$-diagonal is developable.*

PROOF. Let X be a space with a $\overline{G}_\delta$-diagonal and a strict feathering in some compactification bX. By Propositions 1.2 and 1.3, there is a sequence $\{U_n \mid n = 1, 2, \cdots\}$ of neighborhoods of Δ_X, open in $bX \times bX$, satisfying

(1) $\cap \{U_n[x] \mid n = 1, 2, \cdots\} \subset X$, for each $x \in X$; and

(2) for each $x \in X$ and each n, there is an integer m such that $\text{Cl}_{bX}(U_m[x]) \subset U_n[x]$.

Without loss of generality, we may assume that $U_{n+1} \subset U_n$, for each n.

Since X has a $\overline{G}_\delta$-diagonal, it follows from Proposition 1.4 that there is a sequence $\{W_n \mid n = 1, 2, \cdots\}$ of neighborhoods of Δ_X, open in $X \times X$, satisfying

(3) $\Delta_X = \cap \{W_n \mid n = 1, 2, \cdots\}$; and

(4) for each $x \in X$ and each n, there is an integer m such that $\text{Cl}_X(W_m[x]) \subset W_n[x]$.

For each n, let G_n be an open subset of $bX \times bX$ such that $G_n \cap (X \times X) = W_n$. Without loss of generality, we may assume that $G_{n+1} \subset G_n$, for each n.

Now, for each n, let $V_n = U_n \cap G_n$, and note that $V_{n+1} \subset V_n$. To show that X is developable, it suffices to prove that $\{V_n[x] \mid n = 1, 2, \cdots\}$ is a local base (in bX) at x, for each $x \in X$. It will then follow immediately that $\{V_n \cap (X \times X) \mid n = 1, 2, \cdots\}$ satisfies the conditions of Proposition 1.1.

Let $x \in X$ and H be any neighborhood of x in bX. We will show that $V_n[x] \subset H$, for some n. First we will prove that, for $y \in bX - H$, there is an integer $k(y)$ such that $y \notin \text{Cl}_{bX}(V_{k(y)}[x])$.

Case (i). If $y \notin X$, then by (1) there is an integer n such that $y \notin U_n[x]$. By (2) there is an integer m such that $\text{Cl}_{bX}(U_m[x]) \subset U_n[x]$. Thus $y \notin \text{Cl}_{bX}(U_m[x])$, and consequently $y \notin \text{Cl}_{bX}(V_m[x])$. In this case, take $k(y) = m$.

Case (ii). If $y \in X - H$, then $y \neq x$, so $(x, y) \in (X \times X) - \Delta_X$. By (3), $(x, y) \notin W_n$ for some n, so $y \notin W_n[x]$. By (4), there is an integer m

such that $\mathrm{Cl}_X(W_m[x]) \subset W_n[x]$. Hence $y \notin \mathrm{Cl}_X(W_m[x])$. Now $W_m = G_m \cap (X \times X)$, so $W_m[x] = G_m[x] \cap X$. Thus

$$\mathrm{Cl}_X(W_m[x]) = \mathrm{Cl}_X(G_m[x] \cap X) = \mathrm{Cl}_{bX}(G_m[x] \cap X) \cap X$$
$$= \mathrm{Cl}_{bX}(G_m[x]) \cap X,$$

since X is dense in bX and $G_m[x]$ is open in bX. Thus we have $y \in X$, but $y \notin \mathrm{Cl}_X(W_m[x])$, so $y \notin \mathrm{Cl}_{bX}(G_m[x])$, and hence $y \notin \mathrm{Cl}_{bX}(V_m[x])$. In this case, take $k(y) = m$.

Now $\{bX - \mathrm{Cl}_{bX}(V_{k(y)}[x]) \mid y \in bX - H\}$ is a cover of $bX - H$ by sets open in bX. By compactness of $bX - H$, there is a finite subset $F \subset bX - H$ such that $\{bX - \mathrm{Cl}_{bX}(V_{k(y)}[x]) \mid y \in F\}$ covers $bX - H$. Then $\cap \{V_{k(y)}[x] \mid y \in F\}$ is a subset of H, containing x and open in bX. Since $V_{n+1} \subset V_n$, for each n, we have $V_{n+1}[x] \subset V_n[x]$, for each n and each $x \in X$. Thus $\cap \{V_{k(y)}[x] \mid y \in F\} = V_n[x]$, where $n = \max\{k(y) \mid y \in F\}$. Hence $x \in V_n[x] \subset H$. $\blacksquare$

Combining the results of 2.1, 2.2, and 2.3, we obtain

THEOREM 2.4. *A completely regular space is developable iff it is a strict p-space with a $\overline{G}_\delta$-diagonal.*

3. **θ-refinable spaces.** In the light of 2.4 it would be of interest to know under what conditions a p-space with a G_δ-diagonal is a strict p-space with a $\overline{G}_\delta$-diagonal. M. M. Čoban has shown that two such conditions are metacompactness [9] and subparacompactness (also called σ-paracompactness) [10]. Theorem 3.3 below generalizes both of these results.

A space X is *θ-refinable* iff, for every open cover γ of X, there is a sequence $\{\gamma_n \mid n = 1, 2, \cdots\}$ of open refinements of γ such that, for each $x \in X$, there is an integer m such that X is in at most a finite number of elements of γ_m.

The concept of θ-refinability was introduced by Worrell and Wicke [13], who also proved that all developable spaces are θ-refinable. It is clear that metacompact spaces are θ-refinable and D. K. Burke [7] has shown that all subparacompact spaces are θ-refinable.

THEOREM 3.1. *A regular θ-refinable space with a G_δ-diagonal has a $\overline{G}_\delta$-diagonal.*

PROOF. Let X be a regular, θ-refinable space with a G_δ-diagonal. Then there is a sequence $\{\mu_n \mid n = 1, 2, \cdots\}$ of open covers of X such that for each $x \in X$, $\cap \{\mathrm{St}(x, \mu_n) \mid n = 1, 2, \cdots\} = \{x\}$. By regularity, there is an open cover γ_1 of X such that $\bar{\gamma}_1 = \{\overline{G} \mid G \in \gamma_1\} < \mu_1$. Since

X is θ-refinable there is a sequence $\{\lambda_m^1 \mid m = 1, 2, \cdots\}$ of open refinements of γ_1 such that, for each $x \in X$, there is an integer m such that x is in at most a finite number of elements of λ_m^1.

Now let $n > 1$, and assume that, for $1 \leq i \leq n-1$, we have defined covers γ_i and $\{\lambda_m^i \mid m = 1, 2, \cdots\}$. By regularity, there is an open cover γ_n of X such that $\bar{\gamma}_n < \mu_n \wedge \bigwedge \{\lambda_k^m \mid 1 \leq m \leq n-1 \text{ and } 1 \leq k \leq n-1\}$. Since X is θ-refinable, there is a sequence $\{\lambda_m^n \mid m = 1, 2, \cdots\}$ of open refinements of γ_n such that, for each $x \in X$, there is an integer m such that x is in at most a finite number of elements of λ_m^n.

Finally, define $\Lambda = \{\lambda_m^n \mid n = 1, 2, \cdots \text{ and } m \leq n\}$. Clearly Λ is a countable family of open covers of X. We will show that Λ satisfies the conditions for a $\overline{G}_\delta$-diagonal. There is a natural way of ordering the elements of Λ so that they form a sequence; $(\lambda_1^1, \lambda_1^2, \lambda_2^2, \lambda_1^3, \lambda_2^3, \cdots)$.

For each $x \in X$,

$$\bigcap\{\mathrm{St}(x, \lambda_m^n) \mid \lambda_m^n \in \Lambda\} \subset \bigcap\{\mathrm{St}(x, \gamma_n) \mid n = 1, 2, \cdots\}$$
$$\subset \bigcap\{\mathrm{St}(x, \mu_n) \mid n = 1, 2, \cdots\} = \{x\}.$$

This follows from the fact that each λ_m^n refines $\bar{\gamma}_n$ and $\bar{\gamma}_n$ refines μ_n. Thus part (a) of the definition is satisfied. To verify part (b), let $x \in X$ and $\lambda_m^n \in \Lambda$. By the definition of Λ, $m \leq n$ and $\bar{\gamma}_{n+1} < \lambda_m^n$. Now there is an integer k such that x is in at most a finite number of elements of λ_k^{n+1}. Then

$$\mathrm{Cl}(\mathrm{St}(x, \lambda_k^{n+1})) = \mathrm{Cl}(\bigcup\{U \mid U \in \lambda_k^{n+1} \text{ and } x \in U\})$$
$$= \bigcup\{\overline{U} \mid U \in \lambda_k^{n+1} \text{ and } x \in U\}$$
$$\subset \mathrm{St}(x, \bar{\gamma}_{n+1}) \subset \mathrm{St}(x, \lambda_m^n).$$

If $k \leq n+1$, then $\lambda_k^{n+1} \in \Lambda$ and the proof is complete. If $k > n+1$, then $\lambda_1^{k+1} \in \Lambda$ and $\lambda_1^{k+1} < \gamma_{k+1} < \lambda_k^{n+1}$. Hence

$$\mathrm{Cl}(\mathrm{St}(x, \lambda_1^{k+1})) \subset \mathrm{Cl}(\mathrm{St}(x, \lambda_k^{n+1})) \subset \mathrm{St}(x, \lambda_m^n). \quad \blacksquare$$

The following result is proved by Burke in [7].

THEOREM 3.2. *A θ-refinable p-space is a strict p-space.*

THEOREM 3.3. *A completely regular space is developable iff it is a θ-refinable p-space with a G_δ-diagonal.*

PROOF. Necessity is clear. Sufficiency follows from Theorems 3.1, 3.2, and 2.3. $\quad \blacksquare$

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