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PARALLELIZABILITY REVISITED[1]

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ABSTRACT. A classical theorem (Antosiewicz and Dugundji) states that a dynamical system on a locally compact separable metric space is parallelizable if and only if it is dispersive. In this paper it is shown that separability may be omitted, and, under a further condition, local compactness weakened to local Lindelöfness. The crucial step consists in a purely topological characterization of complete instability.

1. Introduction. The well-known Antosiewicz-Dugundji theorem on parallelizability ([4, Theorem 3]; also see [8, first theorem in 2.4]) reads thus: a dynamical system on a locally compact separable metric space is parallelizable if and only if it is dispersive. The aim of the present paper is to obtain a more general theorem of the same type; one of its corollaries shows that separability may be omitted entirely. It is also shown how local compactness may be replaced by local Lindelöfness, thereby obtaining one of the few theorems in dynamical system theory which apply, e.g., to separable normed linear spaces.

Our principal results are corollaries to Theorem 7. §1 is introductory, §3 has the function of an appendix. The question of parallelizability is first reduced to the purely topological problem of finding cross-sections for fiber bundles (Theorem 6 and Lemma 5). An application of known results then yields Theorem 7. (The method just described is not novel: one may recognize the proof of Theorem 3 in [4] as a special case of the construction of cross-sections for fiber bundles [11, 12.2]; an improvement of the latter has led to a generalization of the former.)

Our terminology and notation follows [1, Chapter I]; part will be recapitulated for the convenience of readers. If π is a dynamical system on a topological space X, then the value of π at $(x, t) \in X \times R^1$ is written as $x\pi t$ (thus the axioms are that $\pi: X \times R^1 \to X$ is con-

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tinuous, $x\pi 0=x$, $(x\pi t)\pi s=x\pi(t+s))$; similarly for $M\pi T$ where $M\subset X$, $T\subset R^1$. A subset $M\subset X$ is called invariant iff $M\pi R^1=M$.

Given a point $x\in X$, we define $C_x=x\pi R^1$ (the trajectory or orbit, through x), $K_x=\overline{C}_x$ (orbit-closure); next we define L_x, D_x, J_x (limit set, prolongation, prolongational limit set) as follows: $y\in L_x$ iff $x\pi t_i\to y$ for some $|t_i|\to\infty$; $y\in D_x$ iff $x_i\pi t_i\to y$ for some $x_i\to x$, $t_i\in R^1$; $y\in J_x$ iff $x_i\pi t_i\to y$ for some $x_i\to x$, $|t_i|\to\infty$.

Obviously

$$\begin{array}{ccccc} C_x & \subset & K_x & \subset & D_x \\ & & \cup & & \cup \\ & & L_x & \subset & J_x \end{array} \tag{1}$$

and, if X is Hausdorff, $K_x=C_x\cup L_x$, $D_x=C_x\cup J_x$.

A point $x\in X$ is called Poisson unstable, divergent, wandering, dispersive iff, respectively,

$$x\notin L_x,\quad L_x=\varnothing,\qquad x\notin J_x,\quad J_x=\varnothing.$$

Iff this holds for all points $x\in X$, then the entire system is termed Poisson unstable, divergent, completely unstable (=almost dispersive in [**10**]), dispersive, respectively. Obvious relations between these concepts follow from (1).

A dynamical system π on X is called parallelizable iff there is a homeomorphism between X and a space of the form $Y\times R^1$, such that, whenever x maps into (y, s), also $x\pi t$ maps into $(y, s+t)$ for all t. Or equivalently [**4**], there exists a global section S for π: a subset $S\subset X$ such that, for every $x\in X$, there is $x\pi\theta\in S$ for a unique $\theta\in R^1$, and the mapping $x\mapsto\theta$ is continuous (then we may take $Y=S$).

Let π be a dynamical system on X. The relation C of being on the same trajectory (i.e., xCy iff $x\in C_y$) is an equivalence relation on X; the equivalence classes are precisely the trajectories. The set of equivalence classes modulo C will standardly be endowed with the quotient topology, denoted by X/C, and called the orbit space (of π). The canonical quotient map $X\to X/C$ will consistently be denoted by $e:X\to X/C$; thus

$$e(x)=C_x\in X/C\quad\text{for } x\in X.$$

1. Lemma. *$e:X\to X/C$ is a continuous open surjection.*

Proof. The quotient map e is a continuous surjection. For any open $G\subset X$, the set

$$e^{-1}(e(G))=G\pi R^1=\bigcup\{G\pi t:\ t\in R^1\}$$

is the union of sets $G\pi t$ each of which is open; indeed, it is the image of G under the homeomorphism $X \approx X$ defined by $x \mapsto x\pi t$.

EXAMPLE. We reproduce here an example, due to Bebutov (see [8, 2.4]), which provides considerable insight into some of the finer points involved. Begin with the parallel system on R^2, with trajectories horizontal (e.g. $\dot{x}=1$, $\dot{y}=0$). Choose a sequence of points $z_n=(x_n, y_n)\in R^2$ with $x_n \to +\infty$, $0<y_n\to 0$ monotonically. Introduce critical points at all z_n (e.g. by modifying the system suitably, but only in ϵ_n-neighborhoods of the z_n); finally omit the negative half-rays $(-\infty, x_n]\times\{y_n\}$. There results a dynamical system π on a space X, with the following properties. X is metrizable and a countable union of compact sets; X/C is Hausdorff and a countable union of compact sets (hence Lindelöf); π is dispersive, but not parallelizable. The Antosiewicz-Dugundji theorem traces this to absence of local compactness of X. However, it is Lindelöf paracompact. Theorem 7 then shows that X/C is not regular, and hence not paracompact; indeed, it is obvious that π does not have the regularity property described below, and this is a necessary condition for parallelizability. It may also be noted that X/C is a connected second-axiom space, which, at all points save one, is locally a 1-manifold (hence locally compact).

In [8, 2.5, 4] Nemyckiĭ conjectured that a dispersive system in a separable complete metric space is parallelizable. Actually the example disproves this: the phase space X is completely metrizable, since it is a G_δ set in R^2. Corollary 10 will show that, if the regularity property is imposed, then completeness is irrelevant.

2. **Completely unstable systems and bundle spaces.**

DEFINITION. A dynamical system has the regularity property iff every invariant neighborhood of any point contains a closed invariant neighborhood of the point.

We list for future reference two easily proved results.

2. LEMMA. *Let π be a system on a regular phase space X. Then*
1. *π has the regularity property iff the orbit space X/C is regular.*
2. *If π is parallelizable, then it has the regularity property.*

We list some further instances of systems with regularity (also see the next lemma). If the phase space X is locally compact and if the orbit space X/C is Hausdorff (equivalently, $C_x=D_x$ for all $x\in X$; see also [7, Theorem 2]), then π has the regularity property; to see this merely use Lemma 1. A system uniformly stable relative to a uniformity for X has the regularity property, since X/C is then completely regular.

3. LEMMA. *Let X be Hausdorff. If π is completely unstable with the regularity property, then it is dispersive, and X/C is Hausdorff. If X is locally compact, then, conversely, dispersiveness implies the regularity property; in non-locally-compact X, it need not.*

PROOF. First, let the system be completely unstable. Then it is divergent, i.e., $L_y=\emptyset$ for all $y\in X$ (indeed, quite generally, $x\in J_x$ for all $x\in L_y$). If the system is not dispersive, and $y\in J_x$, then $y\notin C_x$ from complete instability; thus $\overline{C}_y=C_y\cup L_y=C_y$ is disjoint from C_x. In other words, $U=X-C_y$ is an open invariant neighborhood of C_x. With the regularity property we would have a closed invariant neighborhood $V\subset U$; but then $J_x\subset D_x\subset V$ contradicts $J_x\ni y\notin U$.

Conversely, consider a dispersive system in a locally compact space; we wish to verify the regularity property of π. Take any invariant neighborhood U of any given point x; there is a smaller compact neighborhood V, necessarily with $V\pi R^1\subset U\pi R^1\subset U$. Our result now follows from the observation that, in dispersive systems, $V\pi R^1$ is closed whenever V is compact (actually this is necessary and sufficient for dispersiveness).

Finally, return to the example described earlier; the system is dispersive and obviously not regular.

4. LEMMA. *In a Tichonov phase space, a point x is wandering (i.e., $x\notin J_x$) if and only if there exists a local section S containing x such that $S\pi R^1$ is open and on it the system is parallelizable (or dispersive).*

PROOF. That for wandering points x such a local section exists is Lemma 3 in [4] (also see [12, Theorem 3]), in case the phase space is locally compact and metrizable. For our situation use the generalization of the Whitney-Bebutov theorem (existence of local sections) applying to Tichonov spaces [5, VI, 2.12]. Then on $S\pi R^1$ our system reduces to the parallel system over $S\times R^1$.

For the converse assertion, let U be an invariant neighborhood of x on which the system is dispersive; we may take U open. Now relativize the original system to U; then the prolongational limit set of the relativized system has

$$\emptyset = J_U(x) = J(x) \cap U,$$

so that $x\in U$ cannot belong to $J(x)$.

5. LEMMA. *Let π be a system on a space X. Every global section for π is a cross-section to the quotient map $e:X\to X/C$; if π is completely unstable and X Hausdorff, the two concepts coincide.*

PROOF. Recall that a cross-section to a quotient map $f:X\to Y$ is the range of a continuous map $s: Y\to X$ such that $f\circ s$ is the identity map of Y.

If S is a global section for π, then S is evidently homeomorphic to X/C, whereupon $s:X/C\to X$ may be taken as the inverse map.

For the converse assertion, let π be completely unstable, X Hausdorff, $s:X/C\to X$ continuous, $e\circ s=\text{identity}$, $S=\text{range } s$. Thus S is closed; furthermore, each trajectory $C_x=e^{-1}(e(x))$ intersects S at one point precisely (namely, at $s\circ e(x)$). Since π is completely unstable and hence nonperiodic, there exists a mapping $\theta:X\to R^1$ such that $\theta(x)$ is the unique element in R^1 with $x\pi\theta(x)\in S$. It remains to show that θ is continuous, i.e., that

$$x_i\pi t_i \in S \ni x\pi t, \qquad x_i \to x$$

imply $t_i\to t$. We have

$$s\circ e(x_i) = x_i\pi t_i \to x\pi t = s\circ e(x)$$

from continuity. Let t' be any accumulation point of the t_i. Then $t' = \infty$ is excluded by complete instability and $x_i\to x$, $x_i\pi t_i\to x\pi t\in C_x$; and for finite t', closedness of S yields $t'=t$. Thus indeed $t_i\to t$.

6. THEOREM. *For a system π on a Tichonov space X, the following conditions are mutually equivalent*:

1. *π is completely unstable.*
2. *$e:X\to X/C$ is the projection of a fiber bundle with fiber R^1.*

Part of the proof parallels that of Theorem 3 in [7]. However, there are so many points of difference (differentiability and special phase spaces in [7]) that it seemed advisable to give our proof at length.

PROOF. Evidently $2\Rightarrow 1$: apply Lemmas 4 and 5, noting that e is trivial over slicing neighborhoods, and so has cross-sections there.

For $1\Rightarrow 2$ we wish to show that, for each $x\in X/C$ there is a neighborhood U and a homeomorphism $h_U: U\times R^1\to e^{-1}(U)$ such that the composition $e\circ h_U$ is the projection of $U\times R^1$ on the first factor. Let $e(x')=x$; since π is completely unstable, x' is wandering, so there is a local section S containing x' with $U'=S\times R^1$ open in X and parallelizable. Then $U=e(U')$ is open since e is open, and $x\in U$. Letting $f: U\to S$ be the inverse of $e|S$, the map $(y, t)\mapsto f(y)\pi t$ is a homeomorphism $h_U: U\times R^1\to U'=e^{-1}(U)$ satisfying the required condition.

Our main results are corollaries of the following basic theorem.

7. THEOREM. *Let π be a dynamical system on a Tichonov space X, and assume that*

(*) $$X/C \quad \textit{is paracompact.}$$

Then π is parallelizable if and only if it is completely unstable.

PROOF. Since parallelizable systems are completely unstable, we need only prove one implication. Let π be completely unstable. According to Theorem 6, $e: X \rightarrow X/C$ is the projection of a fiber bundle with R^1 as fiber. From the assumptions, both X and X/C are regular, so that π has the regularity property (Lemma 2). If follows easily that X/C is a T_1 space, and hence a Hausdorff space. Thus we have a fiber bundle with base space paracompact Hausdorff, and fiber R^1, an absolute retract for normal spaces. According to [11] (p. 218, referring to [6]), there exists a cross-section to e; from Lemma 5, this is then a global section for π, and so π is parallelizable.

We will show later (Corollary 14) that (*) is satisfied if X is paracompact and locally Lindelöf, and π has the regularity property. Thus we have

8. COROLLARY. *If π is a dynamical system on a Hausdorff paracompact locally Lindelöf space, then π is parallelizable if and only if it is completely unstable and has the regularity property.*

9. COROLLARY. *If π is a dynamical system on a Hausdorff paracompact locally compact space, then π is parallelizable if and only if it is dispersive.*

PROOF. Lemma 3 reduces this to Corollary 8; local compactness implies local Lindelöfness.

10. COROLLARY. *If π is a system on a metrizable locally separable space, then π is parallelizable if and only if it is completely unstable and has the regularity property.*

11. COROLLARY. *On a space X with metric ρ, let π be a system which is Liapunov stable in the sense that, for every $\epsilon > 0$, there exists $\delta > 0$ with*

$$\rho(x\pi t, y\pi t) < \epsilon \quad \textit{whenever } \rho(x, y) < \delta.$$

Then π is parallelizable if and only if it is Poisson unstable.

PROOF. For Liapunov stable systems we have $L_x = J_x$, so that Poisson instability is equivalent to complete instability. The assertion now follows from Theorem 7, since X/C is metrizable. Indeed, there exists an equivalent metric d on X such that

$$d(x\pi t, y\pi t) = d(x, y) \quad \text{for all } x, y \text{ in } X, \ t \in R^1,$$

see [1, IV, 2.8]. It is then easily seen that

$$\inf\{d(x\pi t, y): t \in R^1\} = \inf\{d(x\pi t, y\pi s): t, s \in R^1\},$$

which obviously defines a metric on X/C.

3. Some topological results. The following assertion is classical (for Hausdorff spaces, [2, I, §9.10], [3, XI, 7.3]).

12. PROPOSITION. *A regular locally compact space is paracompact if and only if it is the direct sum of regular locally compact σ-compact spaces.*

13. PROPOSITION. *A regular locally Lindelöf space is paracompact if and only if it is the direct sum of regular Lindelöf spaces.*

This proof is modelled on that of the preceding assertion, making use of the following two properties: every Lindelöf subset meets at most countably many members of any locally finite collection; the countable union of Lindelöf subsets is Lindelöf. It may be noted that here paracompactness can be replaced by the following property ("σ-paracompactness"): every open cover can be refined by a locally countable cover. Thus for regular locally Lindelöf spaces, σ-paracompactness implies paracompactness.

14. COROLLARY. *Let $p: X \to Y$ be the projection of a bundle space, and assume that X is paracompact and locally Lindelöf, Y is regular, the fiber F is connected. Then Y is paracompact.*

PROOF. According to Proposition 13, X is the direct sum of regular Lindelöf spaces. Since the fibers are connected and $p: X \to Y$ is an open map, Y also decomposes into a direct sum of spaces Y_i, whereupon the restrictions $p|X_i: X_i \to Y_i$ of p are continuous open surjections. Thus each Y_i is Lindelöf (and regular), so that their direct sum Y is paracompact according to Proposition 13.

In §2 it is apparent how much depends on paracompactness of X/C (note also that this is necessary for existence of cross-sections if X is paracompact); in Corollaries 8 to 10, the added requirements needed to ensure this were rather coarse. Much stronger results could be obtained if the following were settled:

CONJECTURE. Let $p: X \to Y$ be the projection of a fiber bundle with R^1 as fiber (or merely a Lindelöf fiber), and assume that Y is regular. If X is paracompact then so is Y.

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