

THE CONSTRUCTION OF A $\bar{\partial}$ -SIMPLE COVERING¹

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ABSTRACT. Let M be a complex manifold. It is shown by simple means that an arbitrary fine open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of M exists such that for every form ω of class C^∞ and bidegree (p, q) with $\bar{\partial}\omega = 0$ on $U_{i_0} \cap \cdots \cap U_{i_p}$ there exists a form ψ of class C^∞ on $U_{i_0} \cap \cdots \cap U_{i_p}$ such that $\bar{\partial}\psi = \omega$ provided $q \geq 1$.

Let $A^{p,q}(U)$ be the vector space of forms of bidegree (p, q) and class C^∞ on the open set U of the complex manifold M . The exterior derivative d splits into $d = \partial + \bar{\partial}$, where ∂ has bidegree $(1, 0)$ and $\bar{\partial}$ has bidegree $(0, 1)$. The open subset U is said to be $\bar{\partial}$ -acyclic if and only if for every form $\omega \in A^{p,q}(U)$ with $\bar{\partial}\omega = 0$ with $q \geq 1$ and $p \geq 0$, a form $\psi \in A^{p,q-1}(U)$ exists such that $\omega = \bar{\partial}\psi$. An open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of M is said to be $\bar{\partial}$ -simple if and only if each finite, nonempty intersection $U_{i_0} \cap \cdots \cap U_{i_p} \neq \emptyset$ is $\bar{\partial}$ -acyclic. An open Stein subset U of M is $\bar{\partial}$ -acyclic. The intersection of finitely many open Stein subsets is a Stein subset. Hence, any covering of M by open Stein subsets is $\bar{\partial}$ -simple. Therefore, the existence of a $\bar{\partial}$ -simple covering is assured.

However, this method requires considerable means. The theory of coherent analytic sheaves, cohomology theory with coefficients in a sheaf, Dolbeault's theorem and Cartan's Theorem B on Stein manifolds are needed. Here, a direct proof will be given, which avoids all these tools. The referee informed the author that a similar idea was used by Kenneth Hoffman in his M.I.T. lectures in 1964 to construct an acyclic covering. His proof is not published. The author's proof also arose from classroom needs.

For $a \in \mathbb{C}$ and $r > 0$, define $D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$. Define $D(r) = D(0, r)$ and $D = D(1)$. If $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, define

$$D^m(a, r) = D(a_1, r) \times \cdots \times D(a_m, r).$$

Define $D^m(r) = D^m(0, r)$ and $D^m = D^m(1)$.

The following two theorems are well known. Direct and simple proofs can be found, for instance, in the textbook of Narasimhan [3, pp. 130–138].

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THEOREM 1 (GROTHENDIECK).² D^m is $\bar{\partial}$ -acyclic.

THEOREM 2 (OKA).³ Let $G \neq \emptyset$ be an open subset of \mathbb{C}^m . Suppose that $G \times D$ is $\bar{\partial}$ -acyclic. Let $f: G \rightarrow \mathbb{C}$ be a holomorphic function. Then $G_f = \{z \in G \mid |f(z)| < 1\}$ is $\bar{\partial}$ -acyclic.

If $f: G \rightarrow \mathbb{C}^q$ is holomorphic, and if $f = (f_1, \dots, f_q)$, define

$$G_f = G \cap f^{-1}(D^q) = \{z \in G \mid |f_i(z)| < 1 \text{ for } i = 1, \dots, q\}.$$

LEMMA 3.⁴ Let $G \neq \emptyset$ be open in \mathbb{C}^m . Suppose that $G \times D^n$ is $\bar{\partial}$ -acyclic for every integer $n \geq 0$. Let $f: G \rightarrow \mathbb{C}^q$ be holomorphic. Then $G_f \times D^n$ is $\bar{\partial}$ -acyclic for every integer $n \geq 0$.

PROOF. By assumption the lemma is true for $q=0$. Suppose that it is true for $q-1$. Let $f: G \rightarrow \mathbb{C}^q$ be holomorphic. Then $f = (g, h)$ with $g: G \rightarrow \mathbb{C}^{q-1}$ and $h: G \rightarrow \mathbb{C}$. Then $G_g \times D^n \times D$ is $\bar{\partial}$ -acyclic for every $n \geq 0$. According to Theorem 2,

$$\begin{aligned} (G_g \times D^n)_h &= \{(z, w) \in G_g \times D \mid |h(z)| < 1\} = G_{(g,h)} \times D^n \\ &= G_f \times D^n \end{aligned}$$

is $\bar{\partial}$ -acyclic. q.e.d.

Obviously, $\bar{\partial}$ -acyclic is invariant against biholomorphic maps. An open subset U of M is said to be a biholomorphic polydisk if U can be biholomorphically mapped onto D^m . Hence, any biholomorphic polydisk is $\bar{\partial}$ -acyclic.

THEOREM 4. Let M be a complex manifold of dimension m . Suppose that for each $i=0, 1, \dots, p$ a biholomorphic map $\alpha_i: M \rightarrow M_i$ with $M_i \subseteq \mathbb{C}^m$ and an open subset U_i of M with $\alpha_i(U_i) = D^m$ are given. If $U = U_0 \cap \dots \cap U_p \neq \emptyset$, then $U \times D^n$ is $\bar{\partial}$ -acyclic for all integers $n \geq 0$.

PROOF. Define the biholomorphic map $\alpha_{in}: M \times D^n \rightarrow M_i \times D^n$ by $\alpha_{in}(z, w) = (\alpha_i(z), w)$ for $(z, w) \in M \times D^n$. Then $\alpha_{in}(U_i \times D^n) = D^m \times D^n$.

Each $U_i \times D^n$ is $\bar{\partial}$ -acyclic (Theorem 1). Hence, the theorem is true for $p=0$. Suppose that the theorem is true for $p-1 \geq 0$. Define $G = \alpha_0(U_0 \cap \dots \cap U_{p-1})$. Then

$$G \times D^n = \alpha_{0n}((U_0 \cap \dots \cap U_{p-1}) \times D^n)$$

is $\bar{\partial}$ -acyclic for each $n \geq 0$. The map $f = \alpha_p \circ \alpha_0^{-1}: G \rightarrow \mathbb{C}^m$ is holomorphic. According to Lemma 3,

² See Serre [4, Chapter 18], Dolbeault [2] and Narasimhan [3, Theorem 2.13.5].

³ See Narasimhan [3, Theorem 2.14.7].

⁴ Compare Narasimhan [3, Corollary 2.14.8].

$$\begin{aligned}
\alpha_{0n}^{-1}(G_f \times D^n) &= \alpha_{0n}^{-1}((G \cap f^{-1}(D^m)) \times D^n) \\
&= \alpha_0^{-1}(\alpha_0(U_0 \cap \cdots \cap U_{p-1}) \cap \alpha_0(\alpha_p^{-1}(D^m)) \times D^n) \\
&= (U_0 \cap \cdots \cap U_{p-1} \cap \alpha_p^{-1}(D^m)) \times D^n \\
&= (U_0 \cap \cdots \cap U_{p-1} \cap U_p) \times D^n
\end{aligned}$$

is $\bar{\delta}$ -acyclic for each $n \geq 0$. q.e.d.

This is all the analysis needed. The remainder of the proof uses topological methods only.

LEMMA 5. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ and $\mathfrak{B} = \{V_i\}_{i \in I}$ be locally finite, open coverings of the metrizable, locally compact space M . Suppose that \bar{V}_i is compact and contained in U_i for each $i \in I$. Then it is possible to assign each point x in M an open relative compact neighborhood $B(x)$ such that

$$(B(x) \cup B(y)) \cap \bar{V}_i \neq \emptyset \quad \text{and} \quad B(x) \cap B(y) \neq \emptyset$$

imply $(B(x) \cup B(y)) \subseteq U_i$ if $i \in I$.

PROOF. Take a metric d on M . The distance between two subsets A and B of M shall be denoted by $d(A, B)$. For each $i \in I$, take an open, relative compact subset W_i of M with $\bar{V}_i \subseteq W_i \subseteq \bar{W}_i \subseteq U_i$. For $x \in M$ define

$$\begin{aligned}
B(x, r) &= \{y \mid d(x, y) < r\} \quad (0 < r \in \mathbb{R}), \\
a_i &= \text{dist}(\bar{V}_i, M - W_i) > 0, \quad b_i = \text{dist}(\bar{W}_i, M - U_i) > 0, \\
K(x) &= \{i \in I \mid x \in \bar{V}_i\}, \quad L(x) = \{i \in I \mid x \in \bar{W}_i\}.
\end{aligned}$$

For each $x \in M$, take an open, relative compact neighborhood A_x such that

$$N(x) = \{i \in I \mid \bar{A}_x \cap U_i \neq \emptyset\}$$

is finite. Then $K(x) \subseteq L(x) \subseteq N(x)$. Let C_x^1 be the union of all \bar{V}_i with $i \in N(x) - K(x)$. Let C_x^2 be the union of all \bar{W}_i with $i \in N(x) - L(x)$. Then $C_x = C_x^1 \cup C_x^2$ is compact and does not contain x . Take a number $r(x) > 0$ such that $B(x, r(x)) \subseteq A_x - C_x$ and such that $r(x) < \min(a_i, b_i)$ for all $i \in N(x)$. Then $B(x) = B(x, r(x))$ is an open neighborhood of x with $\bar{B}(x) \subseteq \bar{A}_x$. Hence, $\bar{B}(x)$ is compact.

Now, it shall be proved, that $B(x) \cap \bar{V}_i \neq \emptyset$ implies $\bar{B}(x) \subseteq W_i$. If $i \in I - N(x)$, then $\emptyset \neq B(x) \cap \bar{V}_i \subseteq A_x \cap U_i = \emptyset$. Hence $i \in N(x)$. If $i \in N(x) - K(x)$, then $\emptyset \neq B(x) \cap \bar{V}_i \subseteq (A_x - C_x) \cap C_x = \emptyset$. Hence $i \in K(x)$, which means $x \in \bar{V}_i$. If $y \in \bar{B}(x)$, then $d(x, y) \leq r(x) < a_i$, because $i \in K(x) \subseteq N(x)$. Now $x \in \bar{V}_i$ and $d(x, y) < a_i$ imply $y \in W_i$. Therefore, $\bar{B}(x) \subseteq W_i$.

Similarly, it can be proved, that $B(x) \cap \overline{W}_i \neq \emptyset$ implies $\overline{B}(x) \subseteq U_i$. Suppose that $B(x) \cap B(y) \neq \emptyset$ and $(B(x) \cup B(y)) \cap \overline{V}_i \neq \emptyset$. Without loss of generality, $B(x) \cap \overline{V}_i \neq \emptyset$ can be assumed. Then $\overline{B}(x) \subseteq W_i$ is true. Now, $B(x) \cap B(y) \neq \emptyset$ implies $B(y) \cap \overline{W}_i \neq \emptyset$; therefore $\overline{B}(y) \subseteq U_i$. Together, $\overline{B}(x) \cup \overline{B}(y) \subseteq W_i \cup U_i = U_i$ is proved. q.e.d.

A complex manifold is assumed to be pure dimensional and to have a countable base of open sets.

LEMMA 6. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of the m -dimensional, complex manifold M . Then locally finite, open coverings $\mathfrak{B} = \{B_j\}_{j \in J}$ and $\mathfrak{C} = \{C_j\}_{j \in J}$ and a family $\mathfrak{A} = \{\alpha_j\}_{j \in J}$ of biholomorphic maps exist satisfying the following conditions:

1. The map $\alpha_j: A_j \rightarrow A'_j$ maps the open subset A_j of M biholomorphically onto an open subset A'_j of \mathbb{C}^m .
2. A refinement map $\tau: J \rightarrow I$ exists such that $C_j \subset \overline{C}_j \subset B_j \subset \overline{B}_j \subset A_j \subseteq U_{\tau(j)}$ for each $j \in J$.
3. If $k \in J$ and $j \in J$ with $B_k \cap B_j \neq \emptyset$ then $\overline{B}_k \subseteq A_j$.
4. For each $j \in J$, $\alpha_j(B_j) = D^m$ and $\alpha_j(C_j) = D^m(\frac{1}{2})$.

PROOF. Because every covering on M has an arbitrary small locally finite refinement, it can be assumed, that \mathfrak{U} is already locally finite, and that for each $i \in I$ a biholomorphic map $\beta_i: U_i \rightarrow U'_i$ onto an open subset U'_i of \mathbb{C}^m exists. Moreover, it can be assumed, that each U_i contains the compact closure \overline{V}_i of an open set $V_i \neq \emptyset$ such that $\mathfrak{V} = \{V_i\}_{i \in I}$ is a covering of M . According to Lemma 5, pick an open, relative compact neighborhood $B(x)$ of x such that $B(x) \cap B(y) \neq \emptyset$ and $(B(x) \cup B(y)) \cap \overline{V}_i \neq \emptyset$ imply $(B(x) \cup B(y)) \subseteq U_i$. For each $x \in I$, select an index $\rho(x) \in I$ with $x \in V_{\rho(x)}$. Define $\beta_{\rho(x)}(x) = a(x)$. A number $r_1(x) > 0$ and an open neighborhood $B_1(x)$ of x exists such that

$$x \in B_1(x) \subset \overline{B}_1(x) \subset B(x) \cap V_{\rho(x)} \subset \overline{V}_{\rho(x)} \subset U_{\rho(x)},$$

$$\beta_{\rho(x)}(B_1(x)) = D^m(a(x), r(x)).$$

Then $\overline{B}_1(x)$ is compact. Define $C_1(x) = \beta_{\rho(x)}^{-1}(D^m(a(x), r(x)/2))$. Then $C_1(x)$ is open; $\overline{C}_1(x)$ is compact and contained in $B_1(x)$. A biholomorphic map $\gamma_x: \mathbb{C}^m \rightarrow \mathbb{C}^m$ is defined by

$$\gamma_x(z) = (z - a(x))/r(x).$$

Then $\gamma_x(D^m(a(x), r(x))) = D^m$ and $\gamma_x(D^m(a(x), r(x)/2)) = D^m(\frac{1}{2})$. Hence

$$\delta_x = \gamma_x \circ \beta_{\rho(x)}: U_{\rho(x)} \rightarrow \gamma_x(U'_{\rho(x)})$$

is a biholomorphic map with $\delta_x(B_1(x)) = D^m$ and $\delta_x(C_1(x)) = D^m(\frac{1}{2})$.

For each $i \in I$, finitely many points $x_{i1}, \dots, x_{iq(i)}$ in \bar{V}_i exist such that

$$(1) \quad \bar{V}_i \subseteq C_1(x_{i1}) \cup \dots \cup C_1(x_{iq(i)}).$$

Define $J = \{(i, \mu) \mid i \in I \text{ and } 1 \leq \mu \leq q(i)\}$. Define $\tau: J \rightarrow I$ by $\tau(i, \mu) = \rho(x_{i,\mu})$. If $j = (i, \mu) \in J$, define $C_j = C_1(x_j)$, and $B_j = B_1(x_j)$, and $A_j = U_{\tau(j)}$, and $A'_j = \gamma_{x_j}(U'_{\tau(j)})$, and $\alpha_j = \delta_{x_j}$.

Obviously, the conditions 1, 2, and 4, are satisfied. If $j = (i, \mu)$ and $k = (p, \nu)$ belong to J with $B_k \cap B_j \neq \emptyset$, then

$$\begin{aligned} \emptyset \neq B_k \cap B_j &= B_1(x_{i,\mu}) \cap B_1(x_{p,\nu}) \subseteq B(x_{i,\mu}) \cap B(x_{p,\nu}), \\ x_{i\mu} &\in (B(x_{i\mu}) \cup B(x_{p\nu})) \cap \bar{V}_{\rho(x_{i\mu})}. \end{aligned}$$

Hence $\bar{B}(x_{p\nu}) \subseteq U_{\rho(x_{i\mu})}$ which implies $\bar{B}_k \subseteq U_{\tau(j)} = A_j$. Therefore, also condition 3 is satisfied.

Because of (1), $\mathfrak{C} = \{C_j\}_{j \in J}$ is a covering of M . Hence, $\mathfrak{B} = \{B_j\}_{j \in J}$ is also a covering of M .

Each point $x \in M$, has a neighborhood W_x such that $I_x = \{i \in I \mid W_x \cap U_i \neq \emptyset\}$ is finite. Then

$$J_x = \{(i, \mu) \mid i \in I_x, 1 \leq \mu \leq q(i)\}$$

is finite. If $W_x \cap B_j \neq \emptyset$ with $j = (i, \mu) \in J$, then $W_x \cap B(x_{i\mu}) \neq \emptyset$. Now

$$x_{i\mu} \in (B(x_{i\mu}) \cup B(x_{i\mu})) \cap \bar{V}_i, \quad (B(x_{i\mu}) \cap B(x_{i\mu})) \neq \emptyset$$

imply $B(x_{i\mu}) \subseteq U_i$. Therefore, $W_x \cap U_i \neq \emptyset$, which implies $i \in I_x$ and $j = (i, \mu) \in J_x$. Therefore, the covering \mathfrak{B} is locally finite. Consequently, \mathfrak{C} is locally finite too. q.e.d.

THEOREM 7. *Let \mathfrak{U} be an open covering of the complex manifold M . Then open, locally finite, $\bar{\partial}$ -simple refinements $\mathfrak{B} = \{B_j\}_{j \in \mathfrak{n}}$ and $\mathfrak{C} = \{C_j\}_{j \in \mathfrak{n}}$ of \mathfrak{U} exist, where \bar{C}_j and \bar{B}_j are compact with $\bar{C}_j \subset B_j$ for each $j \in J$.*

PROOF. Take the coverings \mathfrak{B} and \mathfrak{C} as constructed in Lemma 6. Take j_0, \dots, j_p in J with $B_{j_0} \cap \dots \cap B_{j_p} \neq \emptyset$. Then $B_{j_0} \cap B_{j_\mu} \neq \emptyset$. Hence $B_{j_0} \subseteq A_{j_\mu}$ for all $\mu = 0, \dots, p$ and $\nu = 0, \dots, p$. Therefore

$$B_{j_0} \cup \dots \cup B_{j_p} \subseteq A_{j_0} \cap \dots \cap A_{j_p} = A.$$

The biholomorphic map α_{j_μ} maps A onto an open subset of \mathbb{C}^m with $\alpha_{j_\mu}(B_{j_\mu}) = D^m$. According to Theorem 4, $B_{j_0} \cap \dots \cap B_{j_p}$ is $\bar{\partial}$ -acyclic. Hence, \mathfrak{B} is $\bar{\partial}$ -simple. The same argument shows that \mathfrak{C} is $\bar{\partial}$ -simple. q.e.d.

The reason for the construction of \mathfrak{B} and \mathfrak{C} and not \mathfrak{B} alone, lies in the following well-known application:⁵ Let E be a holomorphic vector bundle over the compact manifold E . Let $\mathfrak{D}(E)$ be the sheaf of germs of holomorphic sections in E . Let $A^{p,q}(U, E)$ be the vector space of forms of class C^∞ and of bidegree (p, q) on U with coefficients in E . Then $\bar{\partial}$ extends to $\bar{\partial}_E: A^{p,q}(U, E) \rightarrow A^{p,q+1}(U, E)$. The open subset U is said to be $\bar{\partial}_E$ -acyclic if and only if for every form $\omega \in A^{p,q}(U, E)$ with $\bar{\partial}_E \omega = 0$ and with $q \geq 1$ and $p \geq 0$, a form $\psi \in A^{p,q-1}(U, E)$ exists such that $\omega = \bar{\partial}_E \psi$. An open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of M is said to be $\bar{\partial}_E$ -simple, if and only if each finite, nonempty intersection $U_{i_0} \cap \cdots \cap U_{i_p} \neq \emptyset$ is $\bar{\partial}_E$ -acyclic. Obviously, an open subset U of M is $\bar{\partial}_E$ -acyclic if U is $\bar{\partial}$ -acyclic and $E|_U$ is trivial. Hence a $\bar{\partial}$ -simple, open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ is also $\bar{\partial}_E$ -simple if each $E|_{U_i}$ is trivial.

Now, let \mathfrak{U} be a finite, open covering of the compact complex manifold M such that each $E|_{U_i}$ is trivial. Take the $\bar{\partial}$ -simple refinements \mathfrak{B} and \mathfrak{C} of Theorem 7. They are also $\bar{\partial}_E$ -simple. Hence the refinement homomorphisms

$$H^q(\mathfrak{B}, \mathfrak{D}(E)) \xrightarrow{r} H^q(\mathfrak{C}, \mathfrak{D}(E)) \xrightarrow{s} H^q(M, \mathfrak{D}(E))$$

are isomorphisms. Moreover, $H^q(\mathfrak{B}, \mathfrak{D}(E))$ and $H^q(\mathfrak{C}, \mathfrak{D}(E))$ are Fréchet spaces and r is a compact map. Hence $H^q(\mathfrak{C}, \mathfrak{D}(E))$ is finite dimensional, which implies that $H^q(M, \mathfrak{D}(E))$ also is finite dimensional.

Naturally, in this application, the Čech cohomology with coefficients in a sheaf and Dolbeault's theorem are used, but the theory of coherent analytic sheaves and Cartan's Theorem B are not needed.

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⁵ See Cartan-Serre [1] and Serre [4].