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A PRIMARY DECOMPOSITION FOR TORSION MODULES

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ABSTRACT. A definition of primary module is given and a theorem is proved characterizing rings for which each torsion module, in the sense of S. E. Dickson, decomposes as a direct sum of its primary submodules. This theorem is applied to obtain a generalization of Fuchs' theorem on the additive group structure of Artinian rings.

1. **Introduction.** S. E. Dickson [2], [3] has investigated a primary decomposition for torsion modules over an arbitrary ring and Alin [1] has characterized rings for which this primary decomposition holds. The purpose of this note is to define "primary module" in such a way that the primary decomposition holds for a larger class of rings, in particular, for all Noetherian rings.

All rings R have a unit and modules are unitary left R-modules. If M is an R-module, M^+ denotes the underlying additive group of M. It is well known that if S is a simple R-module, then S^+ is a direct sum of copies of Z_p, the cyclic group of order p, or a direct sum of copies of Q, the additive group of rational numbers. In the first case we say that S is of type p and in the second, S is of type Q.

Let $p_1, p_2, \cdots$ be an indexing of the positive primes and for each $i=1, 2, \cdots$ let $\mathcal{S}_i$ be a representative set of simple R-modules of type p_i. Let $\mathcal{S}_0$ be a representative set of simple R-modules of type Q. For $i=0, 1, \cdots$ let $\mathfrak{J}_i$ be the torsion class generated by $\mathcal{S}_i$ and let $\mathfrak{J}$ be the torsion class generated by $\bigcup_{i=0}^{\infty} \mathcal{S}_i$ [4]. Thus $\mathfrak{J}_i$ (respectively $\mathfrak{J}$) is the class of all modules M such that each nonzero homomorphic image of M has a submodule isomorphic to a member of $\mathcal{S}_i$ (respectively, $\bigcup_{i=0}^{\infty} \mathcal{S}_i$). The classes $\mathfrak{J}, \mathfrak{J}_0, \mathfrak{J}_1, \cdots$ are closed under submodules, direct sums, extensions and homomorphic images. It follows that each module M has a unique largest submodule M_i in $\mathfrak{J}_i$. If $M \in \mathfrak{J}_i$, $i \geqq 1$, M is *p_i-primary* and if $M \in \mathfrak{J}_0$, M is *Q-primary*. The modules $M \in \mathfrak{J}$ are called torsion. The *primary decomposition holds for a ring* R if and only if for each $M \in \mathfrak{J}$, $M = \sum_{i=0}^{\infty} M_i$ (direct), i.e., each torsion module is a direct sum of its primary submodules.

For an R-module M, $\mathrm{Soc}(M)$ denotes the socle of M. We let $T^1(M) = \mathrm{Soc}(M)$ and extend to an ascending chain of submodules $\{T^\alpha(M)\}$

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of M in the usual manner [1]. If $M\in\mathfrak{I}$, $T^{\alpha}(M)=M$ for some ordinal α and the least such ordinal is the *T-length* of M.

For any module M, M_t denotes the usual torsion subgroup of the group M^+ and M_p, for a prime p, denotes the maximum p-primary subgroup of M^+. Note that M_t and M_p are submodules of M.

We use $\mathrm{Ext}(A, B)$ for $\mathrm{Ext}^1_R(A, B)$ and $\mathrm{Hom}(A, B)$ for $\mathrm{Hom}_R(A, B)$. The reader is referred to MacLane [6] for the properties of Ext which are used in what follows.

2. **The main theorem.** The following lemma, in part characterizing the primary submodules of a module M, will be needed in the proof of the main theorem.

LEMMA 2.1. *Let $M\in\mathfrak{I}$. Then*
(1) $M_i=M_{p_i}$ *for* $i\geqq 1$.
(2) $(M_0)^+$ *is a torsion-free divisible group.*

PROOF. (1) Clearly $\mathrm{Soc}(M_i)\subseteq M_{p_i}$ and by induction it is easy to see that $M_i\subseteq M_{p_i}$. But M_i and M_{p_i} are submodules of M so M_{p_i}/M_i is either zero or it has a simple submodule, since $M\in\mathfrak{I}$. The latter choice leads to a contradiction, hence $M_i=M_{p_i}$.

(2) It is clear that $\mathrm{Soc}(M_0)$ is divisible and torsion-free. Assume inductively that $T^{\alpha}(M_0)$ is divisible and torsion-free. Then since

$$T^{\alpha+1}(M_0)/T^{\alpha}(M_0) = \mathrm{Soc}(M_0/T^{\alpha}(M_0))$$

is a direct sum of Q-type simples and since

$$0\rightarrow T^{\alpha}(M_0)\rightarrow T^{\alpha+1}(M_0)\rightarrow T^{\alpha+1}(M_0)/T^{\alpha}(M_0)\rightarrow 0$$

is exact, we get that $T^{\alpha+1}(M_0)$ is torsion-free divisible, since the class of torsion-free divisible groups is closed under extensions. Since $M\in\mathfrak{I}$, $T^{\beta}(M_0)=M_0$ for some β and so M_0 is torsion-free divisible. This completes the proof.

REMARKS. (1) From the previous lemma and the primary decomposition for torsion abelian groups, the primary decomposition holds for any ring which has no Q-type simple modules. In fact, over any ring, if $M\in\mathfrak{I}$ and $M_0=0$, we get $M=\sum_{i=1}^{\infty} M_i$.

(2) For any module M, $\sum_{i\neq j} M_i$ cannot contain a simple from the class $\mathcal{S}_j$ and consequently $\sum_{i=0}^{\infty} M_i$ is always a direct sum.

LEMMA 2.2. *The primary decomposition holds for the ring R if and only if* $\mathrm{Ext}(S, T)=0$ *for each Q-type simple S and each module $T\in\mathfrak{I}$ with $T_0=0$.*

PROOF. The necessity of the condition is clear since if $0\rightarrow T\rightarrow X\rightarrow S\rightarrow 0$ is exact and the primary decomposition holds, we must have $X = X_0 \oplus \sum_{i=1}^{\infty} X_i$ with $X_0 \approx S$ and this implies that the sequence splits.

To prove that the condition is sufficient, let $M \in \mathfrak{I}$. We will prove that $M = \sum_{i=1}^{\infty} M_i$ by showing that $M / \sum_{i=0}^{\infty} M_i$ has no simple submodule. By Lemma 2.1, $\sum_{i=1}^{\infty} M_i = M_t$, so $\sum_{i=1}^{\infty} M_i = M_0 + M_t$. Also by 2.1, M_0 is divisible and so as groups we have

$$\frac{M}{M_t} \approx \frac{M_0 + M_t}{M_t} \oplus \frac{K}{M_t}.$$

Thus any subgroup of $M/M_0 + M_t$ is isomorphic to a subgroup of K/M_t. Hence $M/M_0 + M_t$ contains no p-type simple since K/M_t is torsion-free as a group.

Assume $M/M_0 + M_t$ has a Q-type simple, say $S = X/M_0 + M_t$, $X \subseteq M$. Now $M_0 + M_t/M_0 \approx M_t$ so we have an exact sequence

$$0 \to \frac{M_0 + M_t}{M_0} \approx M_t \to \frac{X}{M_0} \to \frac{X}{M_0 + M_t} = S \to 0.$$

By hypothesis, this sequence must split, so X/M_0 contains a Q-type simple. Thus M/M_0 contains a Q-type simple and this is a contradiction. Hence $M / \sum_{i=0}^{\infty} M_i$ has no simple submodule, so $M = \sum_{i=0}^{\infty} M_i$ and the proof is complete.

THEOREM 2.3. *The primary decomposition holds for the ring R if and only if*

(1) $(\prod_{S \in \mathfrak{C}} S / \sum_{S \in \mathfrak{C}} S)_0 = 0$, *where $\mathfrak{C}$ is a representative set of simples of type p.*

(2) *If $0 \to P \to K \to U \to 0$ is an exact sequence of R-modules with K cyclic, $P \in \mathfrak{I}_i$ for some $i \geqq 1$ and U a Q-type simple, then P has nonlimit ordinal T-length.*

PROOF. To see that the first condition is necessary, suppose U is a Q-type simple contained in the factor module $\prod S / \sum S$. Then we have an exact sequence $0 \to \sum S \to X \to U \to 0$, where $X \subseteq \prod S$. Since the primary decomposition holds, $X \approx U \oplus \sum S$. But then $U \subseteq X \subseteq \prod S$ so

$$0 \neq \mathrm{Hom}(U, U) \subseteq \mathrm{Hom}(U, \prod S) = \prod \mathrm{Hom}(U, S) = 0$$

and we have a contradiction.

If $0 \to P \to K \to U \to 0$ is exact as in (2) above, then $K \approx P \oplus U$ and so P is cyclic. It follows that P has nonlimit ordinal T-length.

To show that (1) and (2) are sufficient, we use Lemma 2.2 and show $\mathrm{Ext}(T, U) = 0$ for U a Q-type simple and $T \in \mathfrak{I}$, $T_0 = 0$.

By previous remarks, $T=\sum_{i=0}^{\infty} T_i$ and applying $\mathrm{Hom}(U, -)$ to the exact sequence

$$0\to\sum T_i\to\prod T_i\to\prod T_i/\sum T_i\to 0$$

we get the exact sequence

$$\mathrm{Hom}(U,\prod T_i/\sum T_i)\to \mathrm{Ext}(U,\sum T_i)\to \mathrm{Ext}(U,\prod T_i).$$

But $\mathrm{Ext}(U, \prod T_i)=\prod \mathrm{Ext}(U, T_i)$ so to show that the condition of Lemma 2.2 holds it is sufficient to prove:

(a) $\mathrm{Hom}(U, \prod T_i/\sum T_i)=0$,

(b) $\mathrm{Ext}(U, T_i)=0$ for $i\geqq 1$.

By (1) and an easy modification of Lemma 2.2 of [2], (a) holds. We prove that (b) holds by showing $\mathrm{Ext}(U, A)=0$ for any p_i-primary module A. The proof is by induction on the T-length of A.

If A is a p_i-primary module of T-length one, then $A=\sum S_\alpha$ where each S_α is a p_i-type simple. As before

$$\mathrm{Hom}(U,\prod S_\alpha/\sum S_\alpha)\to \mathrm{Ext}(U,\sum S_\alpha)\to \mathrm{Ext}(U,\prod S_\alpha)$$

is exact with right end zero since $\mathrm{Ext}(U, S_\alpha)=0$ because U and S_α are simples of different type. Since $p_iU=U$, but $p_i(\prod S_\alpha/\sum S_\alpha)=0$, we must have $\mathrm{Hom}(U, \prod S_\alpha/\sum S_\alpha)=0$. Hence $\mathrm{Ext}(U, A)=0$ if A has T-length one.

Now assume $\mathrm{Ext}(U, A)=0$ for all p_i-primary modules A of T-length $\alpha<\beta$ and let B have T-length β. If

$$(*)\qquad 0\to B\to X\to U\to 0$$

is exact with $B\to X$ the inclusion map, choose $x\in X-B$. Then

$$0\to B\cap Rx\to Rx\to U\to 0$$

is exact, the T-length of $B\cap Rx$ is less than or equal to β and it is not a limit ordinal by (2). Let the T-length of $B\cap Rx$ be $\alpha+1$. Then

$$0\to\frac{B\cap Rx}{T^\alpha(B\cap Rx)}\to\frac{Rx}{T^\alpha(B\cap Rx)}\to U\to 0$$

is exact and since $B\cap Rx/T^\alpha(B\cap Rx)$ has T-length one, the sequence must split. Thus there is a submodule K of Rx containing $T^\alpha(B\cap Rx)$ with $U\approx K/T^\alpha(B\cap Rx)$. Then

$$0\to T^\alpha(B\cap Rx)\to K\to U\to 0$$

is exact and since $T^\alpha(B\cap Rx)$ has T-length $\alpha<\beta$, this sequence must split. Thus K contains a submodule isomorphic to U and so since

$K \subseteq Rx \subseteq X$, X has a submodule isomorphic to U. It follows that the sequence (*) splits and this completes the proof.

3. Applications and examples.

THEOREM 3.1. *Let R be a ring with the property that every maximal left ideal L of R, with R/L a Q-type simple, is finitely generated. Then the primary decomposition holds for R.*

PROOF. We apply Theorem 2.3 and show that conditions (1) and (2) hold.

Let $0 \to P \to K \to U \to 0$ be exact with K cyclic and U a Q-type simple. Then for some left ideals $L \subseteq M$ of R we have $P \approx M/L$ and $U \approx R/M$. But then M is finitely generated so M/L, and hence P, cannot have limit ordinal T-length. Thus (2) holds.

Let $U = R(x_s + \sum S) \subseteq \prod S / \sum S$ where the product and sum are taken over the set $\mathfrak{C}$ as in (1) of 2.3. Let $M = (\sum S : (x_s))$. Then $U \approx R/M$, so $M = Rm_1 + \cdots + Rm_n$ is finitely generated, since U is of type Q. Now for each i, $m_i x_s = 0$ for all but finitely many $S \in \mathfrak{C}$. Hence there is an $x_{s_0} \neq 0$ such that $Mx_{s_0} = 0$. But then $Rx_{s_0} \approx S_0 \approx R/M \approx U$ and this is a contradiction since S_0 is of type p. Hence (1) of 2.3 is satisfied and the proof is complete.

The following corollary generalizes part of Fuchs' Theorem 72.2 [5].

COROLLARY 3.2. *Let R satisfy the hypothesis of* 3.1 *and assume nonzero R-modules have nonzero socles. Then R is the ring direct sum of two sided ideals $R_0, R_1, \cdots, R_n$ where R_0^+ is a direct sum of copies of Q and each R_i^+, $1 \leqq i \leqq n$, is a bounded primary group.*

PROOF. Since nonzero modules have nonzero socles, $R \in \mathfrak{Z}$ and since by 3.1 the primary decomposition holds, we have $R = \sum_{i=0}^{\infty} R_i$. Since R has a unit element, $R = R_0 \oplus \cdots \oplus R_n$. Each of the classes $\mathfrak{Z}_i$ is closed under homomorphic images and right multiplication by elements of R is a left R-homomorphism so each R_i is a two-sided ideal. R_0 is a torsion-free divisible group and so it is a direct sum of copies of Q. Each R_i, $1 \leqq i \leqq n$, is a primary group by Lemma 2.1 and since each R_i is a ring with unit, it must be a bounded group. This proves the corollary.

To construct examples of non-Artinian rings satisfying the hypotheses of Corollary 3.2, let P be an infinite product of copies of Z_p. Define the ring R by $R^+ = P \oplus Z_p$ and $(p_1, i_1)(p_2, i_2) = (i_2 p_1 + i_1 p_2, i_1 i_2)$. Then P is the socle of R and $R/P \approx Z_p$; so nonzero modules have nonzero socles. Since R has no Q-type simples, the hypothesis of 4.1 is clearly satisfied. Since every subgroup of P is an ideal of R, it is clear that R is neither Artinian or Noetherian.

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