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EXTENDING FREE CIRCLE ACTIONS ON SPHERES TO S^3 ACTIONS

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ABSTRACT. Let X be a PL homotopy CP^{2k+1} corresponding by Sullivan's classification to the element $(N_1, \alpha_2, N_2, \cdots, \alpha_k, N_k)$ of $Z \oplus Z_2 \oplus Z \oplus \cdots \oplus Z_2 \oplus Z$.

THEOREM 1. *The topological circle action on S^{4k+3} with orbit space X is the restriction of an S^3 action with a triangulable orbit space iff $\alpha_i = 0$, $i = 2, \cdots, k$; and $N_1 \equiv 0 \bmod 2$; and $\sum (-1)^i N_i = 0$.*

If X admits a smooth structure and satisfies the hypotheses of Theorem 1, a certain smoothing obstruction arising from the integrality theorems vanishes for the corresponding S^3 action.

In this note we establish some necessary conditions for extending smooth free circle actions on homotopy $(4k+3)$-spheres to free S^3 actions (S^3 is the group of unit quaternions). It is known that the orbit space of a free circle action on a homotopy sphere Σ^{4k+3} is a manifold X^{4k+2} with the homotopy type of complex projective space CP^{2k+1}, while the orbit space of an S^3 action is a homotopy quaternionic projective space Y^{4k}. If the circle action is the restriction of the S^3 action (by the maximal torus theorem, the restriction is unique), then X^{4k+2} is an S^2 bundle over Y^{4k}; in fact this bundle is induced by a homotopy equivalence $Y^{4k} \to QP^k$ from the natural fibering $CP^{2k+1} \to QP^k$. In terms of D. Sullivan's classification up to PL isomorphism of PL homotopy complex projective spaces [7] we are able to state in Theorem 1 necessary and sufficient conditions for a homotopy complex projective space X^{4k+2} to fiber in such a way over some PL homotopy quaternionic projective space. We also show that if $k>1$ the PL isomorphism class of the orbit space Y^{4k} of an S^3 action on Σ^{4k+3} is determined by the restricted circle action (the case $k=1$ is of course equivalent to the four dimensional Poincaré conjecture; in this case Theorem 1 implies that only the standard circle action on S^7 extends to an S^3 action).

A smooth circle action satisfying the hypotheses of Theorem 1 will thus extend to a topological S^3 action with a triangulable orbit space.

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It is not known whether additional hypotheses will be necessary to ensure the existence of a smooth extension; as a partial result we define a smoothing obstruction which, although it fails to vanish for many piecewise linear homotopy quaternionic projective spaces, does vanish for orbit spaces of S^3 actions which extend smooth circle actions. In particular, if a smooth circle action on a homotopy S^{11} satisfies the hypotheses of Theorem 1, it is shown that the topological action to which it extends can be smoothed.

The results in §1 formed a portion of my Ph.D. thesis, written under the supervision of Professor G. E. Bredon, to whom I express gratitude.

1. **Fibering homotopy complex projective spaces.** Let E_0 be the canonical 4-disc bundle over QP^k. The circle group acts on E_0 in a fiber preserving manner; let $\overline{E}_0$ denote the orbit space. It is clear that $\overline{E}_0$ is a 3-disc bundle over QP^k with boundary $\partial\overline{E}_0 = CP^{2k+1}$. Recall that if M is a closed smooth manifold, an *h-smoothing* of M is a homotopy equivalence $f: X \to M$ where X is also a closed smooth manifold. We will say that an h-smoothing $f: X \to CP^{2k+1}$ *fibers* if there is an h-smoothing $g: Y \to QP^k$ such that, denoting $g^*\overline{E}_0$ by $\overline{E}_Y$, the induced h-smoothing $\bar{g}: \partial\overline{E}_Y \to \partial\overline{E}_0 = CP^{2k+1}$ is *concordant* to f (i.e. there is a diffeomorphism $c: X \to \overline{E}_Y$ such that $\bar{g}c \simeq f$). Obviously a free circle action on a homotopy S^{4k+3} extends to an S^3 action iff its orbit space fibers.

Fibering is also a sensible notion for an *h-triangulation* (the *PL* analogue of an h-smoothing) of CP^{2k+1}. Given an h-triangulation $g: Y \to QP^k$, let K_Y be a triangulation (in the sense of [6]) of E_Y. Then an h-triangulation $f: X \to CP^{2k+1}$ *fibers* if it is concordant to an h-triangulation $g: \partial K_Y \to \partial\overline{E}_0$ for some h-triangulation $g: Y \to QP^k$. An h-triangulation of CP^{2k+1} fibers when and only when the corresponding topological action of the circle on S^{4k+3} extends to an S^3 action with a triangulable orbit space.

D. Sullivan [7] has classified h-triangulations of CP^n and QP^m up to concordance in the following way. If $f: X \to CP^n$ is an h-triangulation, alter f if necessary to make it transverse regular on $CP^i \subset CP^n$, and let

$$K_i = \operatorname{Ker} f_*: H_i(f^{-1}(CP^i); \boldsymbol{Z}) \to H_i(CP^i; \boldsymbol{Z}).$$

If i is even, the intersection form on K_i is symmetric with index $\tau(K_i)$; put $N_{i/2}(f) = \frac{1}{8}\tau(K_i)$. If i is odd, the intersection form is anti-symmetric; we denote its Arf invariant by $\alpha_{(i+1)/2}(f)$. Sullivan proved that if $n > 2$ the integers $N_j(f)$ and the mod 2 integers $\alpha_k(f)$ are a complete set of invariants for the concordance class of f, and that any pair $(\alpha_2, \cdots, \alpha_p)$; $(N_1, \cdots, N_q)$, with $p=[n/2]$ and $q=[(n-1)/2]$, occurs as the invariants of some h-triangulation of CP^n. The case of QP^m is a little simpler. If $g\colon Y\to QP^m$ is an h-triangulation which is transverse regular on $QP^j\subset QP^m$, put

$$K_i = \operatorname{Ker} g_*\colon H_{2i}(g^{-1}(QP^i)) \to H_{2i}(QP^i), \quad \text{and} \quad M_i(g) = \tfrac{1}{8}\operatorname{index}(K_i).$$

The integers $M_i(g)$ are a complete set of invariants for the concordance class of g if $m>1$. Since $QP^1=S^4$ is a spin manifold, $M_1(g)$ must be even; but any $(m-1)$-tuple $(M_1, \cdots, M_{m-1})$ with M_1 even will occur as invariants of some h-triangulation. To simplify our notation, put $M_0(g)=M_m(g)=0$.

THEOREM 1. *An h-triangulation $f\colon X\to CP^{2k+1}$ fibers iff the following conditions are satisfied*:

(1) $\alpha_i(f)=0, i=2, \cdots, k$.

(2) $N_1(f)$ *is even.*

(3) $\sum_{i=1}^k (-1)^i N_i(f)=0$.

PROOF. We establish first the necessity of condition 1. Thus suppose $g\colon Y\to QP^k$ is an h-triangulation, and identify X with ∂K_Y *via* a concordance. In E_0, CP^{2i-1} bounds the restricted bundle $E_0|QP^{i-1}$. It then follows from [9] that the surgery obstruction to making f h-regular on CP^{2i-1} vanishes. This surgery obstruction is just $\alpha_i(f)$.

The proof is completed by the following

LEMMA. *If $g\colon Y\to QP^k$ is an h-triangulation, and $\bar g\colon\partial K_Y\to\partial\bar E_0 = CP^{2k+1}$ is the induced h-triangulation, then*

$$N_j(\bar g) = M_j(g) + M_{j-1}(g).$$

PROOF. For a PL manifold M, let $P(M)=1+p_1(M)+\cdots$ be the total rational Pontrjagin class. Then, if we put $X=\partial K_Y$ and $l\colon X\to K_Y$ is the inclusion, $\pi\colon K_Y\to Y$ the projection, we have $P(X)=l^*\pi^*[P(Y)g^*P(\bar E_0)]$.

Hirzebruch has shown that $P(\bar E_0)=1+4\rho$, where ρ is the generator of $H^4(QP^k; Z)$ dual to QP^{k-1} [3]. Put $y=g^*\rho\otimes 1\in H^4(Y; Q)$. If ι is the generator of $H^2(CP^{2k+1}; Z)$ dual to CP^{2k}, put $x=\bar g^*\iota\otimes 1\in H^2(X; Q)$, and notice that $l^*\pi^*(y)=x^2$. Therefore,

$$P(X) = l^*\pi^*P(Y)\cdot(1+4x^2). \tag{1}$$

Now suppose $\bar g$ is transverse regular on $CP^{2i}\subset CP^{2k+1}$ and let $V_i = f^{-1}(CP^{2i})$; notice that index $V_i=8N_i(\bar g)+1$.

According to [8] the virtual index formula of Hirzebruch [4, §9] is valid for PL manifolds. Let $L(M)=1+L_1(M)+\cdots$ be the total Hirzebruch class of the manifold M. An application of the virtual index formula to X yields

$$\text{(2)} \qquad \text{index } V_i = 8N_i(\bar{g}) + 1 = \langle (\tanh x)^{2(k-i)+1} L(X), [X] \rangle .$$

From (1) it follows that

$$L(X) = l^*\pi^*L(Y) \frac{2x}{\tanh(2x)};$$

applying a hyperbolic identity,

$$(\tanh x)L(X) = l^*\pi^*L(Y)\cdot(1+\tanh^2 x)\cdot x.$$

Applying (2),

$$\text{(3)} \qquad 8N_i(\bar{g}) + 1 = \langle l^*\pi^*L(Y)(\tanh^2 x)^{k-i}(1+\tanh^2 x)x, [X]\rangle .$$

The Thom class of the bundle E_Y is δx, where $\delta: H^2(X; Q) \to H^3(K_Y, X; Q)$ is the coboundary.

If $[K_Y] \in H_{4k+3}(K_Y, X; Z)$ is an orientation, we may assume $\partial[K_Y] = [X]$ and $\partial x \cap [K_Y] = [Y]$. From (3) it follows that

$$\begin{aligned} 8N_i(\bar{g}) + 1 &= \langle \delta\{l^*\pi^*L(Y)(\tanh^2 x)^{k-i}(1+\tanh^2 x)\cdot x\}, [K_Y]\rangle \\ \text{(4)} \qquad &= \langle l^*\pi^*L(Y)(\tanh^2 x)^{k-i}(1+\tanh^2 x)\cdot \delta x, [K_Y]\rangle \\ &= \langle L(Y)(\tanh^2 \sqrt{y})^{k-i}(1+\tanh^2\sqrt{y}), [Y]\rangle . \end{aligned}$$

The normal bundle of $QP^j \subset QP^k$ is $(k-j)E_0$; its Pontrjagin class is $(1+\rho)^{2k-2j}$. If g is transverse regular to QP^j and $W_j = g^{-1}(QP^j)$ then $h^*P(Y) = P(W_j)\cdot h^*g^*(1+\rho\otimes 1)^{2k-2j}$; $h: W_j \to Y$ being the inclusion. Thus

$$h^*L(Y) = L(W_j)\cdot h^*\left(\frac{\sqrt{y}}{\tanh\sqrt{y}}\right)^{2k-2j}$$

and

$$\begin{aligned} \text{index } W_j &= \langle L(W_j), [W_j]\rangle \\ &= \left\langle L(Y)\cdot\left(\frac{\tanh\sqrt{y}}{\sqrt{y}}\right)^{2k-2j}, y^{k-j}\cap[Y]\right\rangle \\ &= \langle L(Y)\cdot(\tanh^2\sqrt{y})^{k-j}, [Y]\rangle . \end{aligned}$$

Comparing (4),

$$8N_i(\bar{g}) + 1 = \text{index } W_i + \text{index } W_{i-1}.$$

Since index $W_j = 8M_j(g)$ if j is odd and $8M_j(g)+1$ if j is even, $N_i(\bar{g}) = M_i(g) + M_{i-1}(g)$.

REMARK. The lemma shows that the concordance class of g is determined by that of $\bar{g}$ if $k>1$. In terms of S^3 actions, the equivariant homeomorphism class of an S^3 action on S^{4k+3} $(k>1)$ is determined by the circle action to which it restricts. In case $k=1$, Theorem 1 may be applied to the classification of smooth circle actions on S^7 to show that only the standard circle action extends to an S^3 action; see [**5**].

2. **An obstruction to smoothing an h-triangulation.** Consider the case of an h-triangulation $f: M\to S$ of a smooth manifold S. Such an h-triangulation is *concordant to an h-smoothing* if there is an h-smoothing $f': M'\to S$ and a piecewise differentiable homeomorphism $k: M\to M'$ such that $f'k\simeq f$.

Ω_*^{spin} is to be the spin bordism ring. If X is a space, $\Omega_*^{\text{spin}}(X)$ will denote by analogy with [**1**] the spin bordism group of X. Returning to the h-triangulation $f: M\to S$, let $\eta: P\to S$ represent an element of $\Omega_p^{\text{spin}}(S)$. If we replace both M and S by their products with a sphere of high enough dimension, we can assume η is an embedding. Approximate f by a map transverse regular to $\eta(P)$, and put $Q=f^{-1}(\eta(P))$. Then define $\mu_f(\eta)=\langle \hat{A}(Q), [Q]\rangle$, where $\hat{A}=1+\hat{A}_1+\cdots$ is the multiplicative sequence associated with $\frac{1}{2}\sqrt{z}/\sinh\frac{1}{2}\sqrt{z}$.

PROPOSITION. *$\mu_f(\eta)$ depends only on the element of $\Omega_*^{\text{spin}}(S)$ represented by η and the homotopy class of f. If we allow Ω_*^{spin} to act on $\mathbf{Q}$ via $\hat{A}$ then $\mu_f: \Omega_*^{\text{spin}}(S)\to \mathbf{Q}$ is an Ω_*^{spin}-homomorphism. If f is concordant to an h-smoothing, μ_f takes integral values.*

PROOF. The first part follows from a straightforward application of relative transversality and oriented bordism invariance of Pontrjagin numbers; details are omitted. If an embedding $\eta: P\to S$ represents an element of $\Omega_p^{\text{spin}}(S)$ and $\{U\}\in\Omega_m^{\text{spin}}$ then by embedding U in S^l, l large, $\{U\}\cdot\{\eta\}$ is represented by an embedding $U\times P\to S^l\times S$; if f is transverse regular to P then id $\times f: S^l\times M\to S^l\times S$ is transverse regular to $U\times P$, with $(\text{id}\times f)^{-1}(U\times P)=U\times Q$. The second part follows then from the multiplicative property of the $\hat{A}$ genus. Finally, if f is concordant to an h-smoothing then Q may be taken to be a smooth spin manifold. The statement about the integrality of μ_f is then a consequence of the differentiable Riemann-Roch theorem; see [**4**, p. 197].

THEOREM 2. *If $g: Y\to QP^k$ is an h-triangulation such that $\bar{g}: X=\partial K_Y\to CP^{2k+1}$ is concordant to an h-smoothing, then $\mu_g: \Omega_*^{\text{spin}}(QP^k)\to \mathbf{Q}$ takes integral values.*

PROOF. It will suffice to check this on $\Omega_{4*}^{\mathrm{spin}}(QP^k)$ since μ vanishes on components of dimension $\not\equiv 0$ mod 4. Let $\eta: P \to QP^k$ represent an element of $\Omega_{4i}^{\mathrm{spin}}(QP^k)$; assume that η is an embedding. If $p: CP^{2k+1} \to QP^k$ is the natural projection, let $R = p^{-1}(P)$ and notice that R is a spin manifold of dimension $4i+2$. Approximate g by a map transverse regular on P and let $Q = g^{-1}(P)$, $N = \bar{g}^{-1}(R)$, and $K_Q = \pi^{-1}(Q)$, so that $\partial K_Q = N$. Since N admits the structure of a smooth spin manifold, $\langle \exp(c)\hat{A}(N), [N]\rangle$ is integral if c is any element of $H^2(N; \mathbf{Z})$ [4, p. 197]. Since dimension $(N) \equiv 2$ mod 4, $\langle \exp(c)\hat{A}(N), [N]\rangle = \langle \sinh(c)\hat{A}(N), [N]\rangle$.

Now, $P(N) = l^*\pi^*P(Q)(1+4x^2)$ where l, π, and x have been restricted to N, K_Q and N, respectively; compare (1). Therefore $l^*\pi^*A(Q) = A(N)\cdot(\sinh x)/x$, and since $\mu_g(\eta) = \langle \hat{A}(Q), [Q]\rangle$, we have

$$\begin{aligned} \mu_g(\eta) &= \langle \pi^*\hat{A}(Q), [K_Q] \cap \delta x\rangle = \langle \delta(l^*\pi^*\hat{A}(Q)\cdot x), [K_Q]\rangle \\ &= \langle \hat{A}(N)\cdot \sinh x, [N]\rangle \in \boldsymbol{Z}. \end{aligned}$$

In case η is not an embedding, this proof may be modified by replacing each of X, Y, CP^{2k+1}, and QP^k by its product with S^l and approximating η by an embedding of P into $QP^k \times S^l$.

REMARKS. If $I: QP^k \to QP^k$ is the identity map and the $(4k-1)$-skeleton of Y is smooth, then $\mu_g(I)$ is just the Eells-Kuiper invariant [2] of the boundary of a smoothing of $Y_0 = Y-$(open ball). In particular it follows that an h-triangulation of QP^2 is concordant to an h-smoothing iff $u_g(I)$ is integral. For an h-triangulation of QP^3 the necessary and sufficient condition is that $\mu_g(QP^2)$ be integral and $\mu_g(I)$ be *even*. For higher dimensions more complicated smoothing obstructions must be considered; generally u_g mod 2 is an obstruction on components of dimension $\equiv 4$ mod 8 but this refinement may still fail to detect certain nonsmoothable h-triangulations of QP^4.

Finally, we have not solved the problem of extending smooth circle actions on homotopy spheres to smooth S^3 actions for homotopy spheres of dimension greater than seven. In dimension 11, although a topological extension of a smooth circle action can be smoothed if its orbit space is triangulable, it is conceivable that the smoothing would be inconsistent with the circle action.

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