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A CHARACTERIZATION OF HEREDITARILY INDECOMPOSABLE CONTINUA

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ABSTRACT. In this paper a characterization of a hereditarily indecomposable continuum is stated and proved. The motivation for this characterization is a theorem in a recent article by John Jobe.

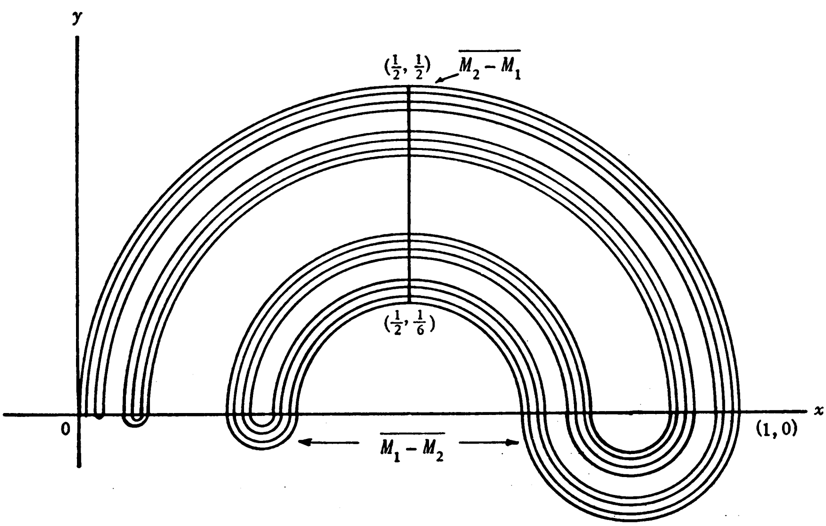
In this paper a characterization of a hereditarily indecomposable continuum is stated and proved. The motivation for this characterization is a theorem proved by Jobe in [1]. This result is as follows:

THEOREM 1. *If M is the 2-finished sum of compact continua, M_1 and M_2, such that M_1 is hereditarily indecomposable and $M_1 \cap M_2 \neq \varnothing$, then there exists at least one point in $M_1 \cap M_2$ which is a limit point of both $(M_1 - M_2)$ and $(M_2 - M_1)$.*

DEFINITION. The set M is the 2-finished sum of continua M_1 and M_2 if $M = M_1 \cup M_2$ and $M_1 - M_2 \neq \varnothing$ and $M_2 - M_1 \neq \varnothing$.

We shall consider the space S to be a Moore space satisfying Axiom 0 and Axiom 1 of R. L. Moore.

First, we suspected that the hypothesis in Theorem 1 was too



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strong. That is, we suspected that M_1 need only be an indecomposable continuum rather than hereditarily indecomposable. The following example in the plane emphasizes the importance of the hypothesis of Theorem 1 as stated. It shows the existence of a compact indecomposable continuum M_1 and a compact continuum M_2 satisfying the hypothesis of Theorem 1 with the exception that M_1 is not hereditarily indecomposable and the conclusion of Theorem 1 is not true.

EXAMPLE 1. Let M_1 be the indecomposable continuum in the plane consisting of those semicircles lying above the x-axis centered at $(1/2, 0)$ with endpoints in the Cantor ternary set on the x-axis together with those semicircles lying below the x-axis with centers at $(1/(6\cdot 3^i), 0)$ and endpoints in the Cantor set. See [3].

Let M_2 be the continuum consisting of M_1 intersected with the closed upper-half plane together with the line segment from $(1/2, 1/6)$ to $(1/2, 1/2)$.

Then $M_1 - M_2$ lies below the x-axis and $M_2 - M_1$ lies above the line $y = 1/6$. Hence there exists no point that is a limit point of both $M_1 - M_2$ and $M_2 - M_1$.

The characterization of a hereditarily indecomposable continuum is in terms of the following defined Property Q. The definition of Property Q is motivated by the condition in Theorem 1.

DEFINITION. Let S be a Moore space and M a continuum in S. Then M has Property Q in S if and only if for every compact continuum N in S such that $N \cap M \neq \varnothing$ and $N \cup M$ is the 2-finished sum of N and M, then there exists a point $p \in M \cap N$ such that p is a limit point of both $M - N$ and $N - M$. A compact continuum M in S has Property Q hereditarily in S if and only if each subcontinuum of M has Property Q in S.

THEOREM 2. *Let T be a Moore space and M a compact continuum in T. Then M is hereditarily indecomposable if and only if for every function f and Moore space S such that f imbeds M in S, then $f(M)$ has Property Q hereditarily in S.*

PROOF. Assume the condition of the theorem. Let M be a compact continuum in a Moore space S and suppose that M is not hereditarily indecomposable. Then there exists a decomposable subcontinuum $M' = H \cup K \subset M$ where H and K are proper subcontinua of M' and $h \in H - K$. Note that since S is a Moore space then $S \times S$ is also a Moore space. Define maps f and g from M to $S \times S$ as $f(m) = (m, h)$ and $g(m) = (h, m)$ for each $m \in M$. Then, since both f and g imbed M in $S \times S$, $f(M') = M' \times \{h\} = M_1$ and $g(M') = \{h\} \times M' = M_2$ are homeomorphic to M'. Let $H_1 = f(H)$ and $K_1 = f(K)$. Then $M_1 = H_1 \cup K_1 \subset f(M)$ is decomposable with $(h, h) \in H_1 - K_1$ since $h \in H - K$.

Also the definitions of f and g imply that $M_1 \cap M_2 = \{(h, h)\} = H_1 \cap M_2$ since $H_1 \subset M'$. Thus $H_1 \cup M_2$ is a continuum in $S \times S$. Furthermore, $(H_1 \cup M_2) - M_1 = M_2 - H_1$, and hence

$$\overline{(H_1 \cup M_2) - M_1} \subset M_2.$$

Also $M_1 - (H_1 \cup M_2) = M_1 - H_1 \subset K_1$ and

$$\overline{M_1 - (H_1 \cup M_2)} \subset K_1.$$

Since K_1 and M_2 are disjoint closed sets, no point is a limit point of both $M_1 - (H_1 \cup M_2)$ and $(H_1 \cup M_2) - M_1$. Hence, $M_1 = f(M')$ does not have Property Q in $S \times S$. Therefore, $f(M)$ does not have Property Q hereditarily in $S \times S$ which is a contradiction. Therefore, M is hereditarily indecomposable.

Conversely assume that M is a compact hereditarily indecomposable continuum in a Moore space T. Let f be any function that imbeds M in a Moore space S and consider $f(M)$. Since each subcontinuum of $f(M)$ is itself hereditarily indecomposable, applying Theorem 1 we see that each subcontinuum of $f(M)$ has Property Q in S. Hence $f(M)$ has Property Q hereditarily in S and the condition of the theorem follows.

We thought that in Theorem 2 the condition "for every function f and Moore space S such that f imbeds M in S, $f(M)$ has Property Q hereditarily in S" could be replaced by the condition "M has Property Q hereditarily in T." To see that this cannot be done the following example exhibits a Moore space T and a decomposable compact continuum M in T such that M has Property Q hereditarily in T. Thus, this example complements the statement of Theorem 2.

EXAMPLE 2. Let S_1 and S_2 be two pseudo-arcs in the plane constructed from $(-1, 0)$ to $(0, 0)$ and $(0, 0)$ to $(1, 0)$ respectively such that $S_1 \cap S_2 = \{(0, 0)\}$. Let $p = (0, 0)$. Let T be the subspace of the plane such that $T = S_1 \cup S_2$. Let H and K be nondegenerate proper subcontinua of S_1 and S_2 respectively such that $H \cap K = \{p\}$. Then $M = H \cup K$ is a decomposable compact continuum that has Property Q hereditarily in T.

Suppose N is a subcontinuum of T such that $N \cap M \neq \emptyset$ and $N \cup M$ is the 2-finished sum of N and M. Clearly $p \in N$ and we may assume that $(N \cap S_1) \subset H$ and $K \subset (N \cap S_2)$. Then $M - N = H - (N \cap S_1)$ and $N - M = (N \cap S_2) - K$. Because H and $N \cap S_2$ are pseudo-arcs, p is a limit point of both $H - (N \cap S_1)$ and $(N \cap S_2) - K$. Since $p \in M \cap N$ it has been verified that M has Property Q in T.

Now if M' is a subcontinuum of M, then either (a) $M' \subset S_1$ or (b) $M' \subset S_2$ or (c) $M' - S_1 \neq \emptyset$ and $M' - S_2 \neq \emptyset$. In cases (a) and (b), M'

is hereditarily indecomposable and hence has Property Q in T. In case (c), M' has Property Q in T by the method used to show that M has Property Q in T. Therefore, M has Property Q hereditarily in T.

Since M is decomposable the example is verified.

There is a natural question suggested by Theorem 2 and Example 2—namely, is the property in question really a property of the embedding or a property of the containing space? More precisely, if M is a compact continuum, S is a Moore space, and f and g are two embeddings of M in S, does $f(M)$ have Property Q hereditarily in S if and only if $g(M)$ has Property Q hereditarily in S? Example 3 gives a negative answer to this question.

EXAMPLE 3. Let T and M be as defined in Example 2. By Theorem 2 there exists a Moore space T_1 and an imbedding function f from M to T_1 such that $f(M)$ does not have Property Q hereditarily in T_1. The space T_1 can be picked such that $T \cap T_1 = \emptyset$. Let g be the imbedding from M to T such that $g(m) = m$, $m \in M$. Example 2 reveals that $g(M)$ has Property Q hereditarily in T.

Let $S = T \cup T_1$ and A be open in S if and only if A is the union of an open set in T and an open set in T_1. Note that S is a Moore space. It follows that f and g can be thought of as imbeddings of M into S and clearly $g(M)$ has Property Q hereditarily in S while $f(M)$ does not have Property Q hereditarily in S.

The authors have been able to find two other characterizations of hereditarily indecomposable continua. These two can be found in [**2**] and [**4**].

BIBLIOGRAPHY

1. John Jobe, *The intersection of indecomposable continua*, Proc. Amer. Math. Soc. **23** (1969), 623–624. MR **40** #877.

2. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36. MR **3**, 315.

3. K. Kuratowski, *Topologie*. Vol. 2, Monografie Mat., Tom 21, PWN, Warsaw, 1961; English transl., Academic Press, New York; PWN, Warsaw, 1969. MR **24** #A2958.

4. William R. Zame, *A characterization of hereditarily indecomposable continua*, Proc. Amer. Math. Soc. **17** (1966), 709–710. MR **33** #3274.

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