GROUPS IN WHICH EACH ELEMENT COMMUTES WITH ITS ENDOMORPHIC IMAGES

R. FAUDREE

ABSTRACT. Let G be a finite group in which each element commutes with its endomorphic images. We will give a counterexample to the conjecture that G is abelian.

Let G be a finite group in which each element commutes with its endomorphic images. We will give a counterexample to the conjecture that G is abelian.

For g and h in G , g and $g^{-1}(h^{-1}gh)$ commute. Thus G satisfies the identical relation $(x, y, x) = 1$. Therefore G satisfies the identical relations $(x, y, z, w) = 1$ and $(x, y, z)^3 = 1$ (see [1, p. 322]). Hence G is nilpotent of class ≤ 3 and if G is a p -group for a prime $p \neq 3$, G is nilpotent of class ≤ 2 . We will exhibit for any prime p a nilpotent class 2 p -group in which each element commutes with all of its endomorphic images.

Let

$$G = \langle a_1, a_2, a_3, a_4 : a_i^{p^2} = 1, (a_i, a_j, a_k) = 1, (1 \leq i, j, k \leq 4) \\ \text{and } (1) (a_1, a_2) = a_1^p, (2) (a_1, a_3) = a_3^p, (3) (a_1, a_4) = a_4^p, \\ (4) (a_2, a_3) = a_2^p, (5) (a_2, a_4) = 1, (6) (a_3, a_4) = a_3^p \rangle.$$

Let $E(G)$ and $A(G)$ denote the endomorphisms and automorphisms of G respectively. Let $EZ(G) = \{\theta \in E(G) : \theta(G) \leq Z(G)\}$ and $AZ(G) = \{\theta \in A(G) : \theta(g)g^{-1} \in Z(G) \text{ for all } g \in G\}$ = central automorphisms of G .

We will prove

THEOREM. $E(G) = EZ(G) \cup AZ(G)$.

COROLLARY. *Each element in G commutes with its endomorphic images.*

LEMMA 1. $|G| = p^8$.

PROOF. Clearly from the defining relations $|G| \leq p^8$. We will construct a group of order p^8 which satisfies the defining relations of G . Let

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$$K = \langle k_i, 1 \leq i \leq 4: k_1^p = k_2^p = 1, k_2^{p^2} = k_4^{p^2} = 1$$

$$\text{and } (k_i, k_j) = 1, 1 \leq i, j \leq 4 \rangle$$

$$\text{and } H = \langle h_1, h_3: h_1^{p^3} = h_3^{p^3} = 1 \text{ and } (h_1, h_3) = h_3^p \rangle.$$

K is an abelian group of order p^6 and H is a nilpotent class 2 group of order p^4 . Let σ_1 and σ_3 be the automorphisms of K determined by $\sigma_1(k_1) = k_1$, $\sigma_1(k_3) = k_3$, $\sigma_1(k_2) = k_1^{-1}k_2$ and $\sigma_1(k_4) = k_4^{-p}k_4$ and $\sigma_3(k_1) = k_1$, $\sigma_3(k_3) = k_3$, $\sigma_3(k_2) = k_2^p k_2$ and $\sigma_3(k_4) = k_3^{-1}k_4$. The subgroup of $A(K)$ generated by σ_1 and σ_3 is an abelian group of exponent p and order p^2 . Let θ be the homomorphism of H into $A(K)$ determined by $\theta(h_1) = \sigma_1$ and $\theta(h_3) = \sigma_3$.

Let $S = [K]_\theta H = \{kh: k \text{ in } K \text{ and } h \text{ in } H\}$ be the semidirect product of K by H determined by θ . Thus $hk = (\theta(h^{-1}))(k)h$. S is a group of order p^{10} and the elements $k_1^{-1}h_1^p$ and $k_3^{-1}h_3^p$ generate a subgroup R of order p^2 in $Z(S)$. Let $T = S/R$, $t_1 = h_1 \cdot R$, $t_2 = k_2 \cdot R$, $t_3 = h_3 \cdot R$ and $t_4 = k_4 \cdot R$. The group T of order p^8 is generated by $\{t_i: 1 \leq i \leq 4\}$ and the t_i satisfy the defining relations of G .

LEMMA 2. $Z(G) = G' = G^p = U_p(G)$ where $U_p(G)$ is the set of elements whose order divides p .

PROOF. $G' = \langle a_i^p, 1 \leq i \leq 4 \rangle = G^p$, $G' \subseteq Z(G)$ and $G' \subseteq U_p(G)$.

Since $|G| = p^8$, $|G'| = p^4$. If $[G:Z(G)] \leq p^3$ then $|G'| \leq p^3$ and if $[G:U_p(G)] \leq p^3$ then $|G^p| \leq p^3$.

For a fixed $g \in G$, let

$$\theta_g: G \rightarrow G$$

$$h \rightarrow (h, g).$$

Since G is nilpotent of class 2, θ_g is a homomorphism. $\text{Ker } \theta_g = C_G(\langle g \rangle)$ and $[G:C_G(\langle g \rangle)] = |\theta_g(G)|$.

LEMMA 3. (i) $C_G(\langle a_1 \rangle) = G' \cdot \langle a_1 \rangle$,

(ii) $C_G(\langle a_2 \rangle) = C_G(\langle a_4 \rangle) = G' \cdot \langle a_2, a_4 \rangle$,

(iii) $C_G(\langle a_3 \rangle) = G' \cdot \langle a_3, a_1 a_4 \rangle$.

PROOF. It is easily seen that the right-hand side is contained in the left-hand side. $\theta_{a_1}(G) = \langle a_1^p, a_2^p, a_4^p \rangle$. Thus $[G:C_G(\langle a_1 \rangle)] = p^3$, which gives the first equality. Likewise, $|\theta_{a_i}(G)| = p^2$ and $[G:C_G(\langle a_i \rangle)] = p^2$ ($2 \leq i \leq 4$).

LEMMA 4. Let n_i ($1 \leq i \leq 4$) be integers. The matrix

$$A = \begin{pmatrix} n_2 & 0 & n_3 & n_4 \\ -n_1 & n_3 & 0 & 0 \\ 0 & -n_2 & n_4 - n_1 & 0 \\ 0 & 0 & -n_3 & -n_1 \end{pmatrix}$$

considered over the prime field of characteristic p has rank 0, 2, or 3.

PROOF. Rank $A \neq 4$ since $\det A = 0$. One can verify that if all 2 by 2 submatrices of A have determinant 0 (mod p), then $n_i \equiv 0 \pmod{p}$, ($1 \leq i \leq 4$). Thus Rank $A \neq 1$.

LEMMA 5. If $\theta \notin A(G)$ and $\theta \in E(G)$ then $\theta \in EZ(G)$.

PROOF. $\theta \notin A(G)$ implies $\theta(h) \in G'$ for some $h \notin G'$. Thus $h = \prod_{i=1}^4 a_i^{n_i}$ with $n_i \not\equiv 0 \pmod{p}$ for at least one i . $\theta(a_i, h) = (\theta(a_i), \theta(h)) = 1$ ($1 \leq i \leq 4$), hence $\langle (a_i, h), 1 \leq i \leq 4 \rangle \subseteq \text{Ker } \theta$.

$$\begin{aligned}
 (a_1, h) &= a_1^{pn_2} a_3^{pn_3} a_4^{pn_4}, \\
 (a_2, h) &= a_1^{-pn_1} a_2^{pn_3}, \\
 (a_3, h) &= a_2^{-pn_2} a_3^{p(n_4-n_1)}, \\
 (a_4, h) &= a_3^{-pn_3} a_4^{-pn_1}.
 \end{aligned}
 \tag{*}$$

The matrix of the powers of a_i^p in (*) is A . Rank $A \geq 2$ since $n_i \not\equiv 0 \pmod{p}$ for some i . Hence $|\text{Ker } \theta \cap G'| \geq p^2$ and there exists $\{h_i: 1 \leq i \leq 4\}$ such that $G = \langle h_1, h_2, h_3, h_4 \rangle$ and $\theta(h_i^p) = \theta(h_i^2) = 1$. Thus for $i=1, 2$, $\theta(h_i) \in G'$ and $(\theta(G))' \subseteq \langle \theta(h_3), \theta(h_4) \rangle'$. Therefore $|\theta(G')| = |(\theta(G))'| \leq p$ and $|\text{Ker } \theta \cap G'| \geq p^3$. We can assume $\theta(h_i^p) = 1$ and $\theta(h_i) \in G'$, $1 \leq i \leq 3$. Hence $(\theta(G))' = \langle 1 \rangle$, $G' \subseteq \text{Ker } \theta$ and $\theta(G) \subseteq G'$. Let

$$\begin{aligned}
 R &= \{g \in G: [G: C_G(\langle g \rangle)] = p^2\}, \\
 R_1 &= \{g \in G: g \notin G' \text{ and } g = a_2^r a_4^s\}, \\
 R_2 &= \{g \in G: g \notin G' \text{ and } g = a_1^r a_3^s a_4^r\}.
 \end{aligned}$$

At least one of the integers r or s is not divisible by p since $g \notin G'$.

LEMMA 6. $R = R_1 \cup R_2$.

PROOF. Let $h = \prod_{i=1}^4 a_i^{n_i}$. $h \in R$ if and only if $|\theta_h(G)| = p^2$. The argument in Lemma 5 yields that $|\theta_h(G)| = p^2$ if and only if Rank $A = 2$. By inspection one can observe that Rank $A = 2$ if $h \in R_1 \cup R_2$. Let d_{ij} denote the (i, j) -minor of A . If Rank $A = 2$ then $d_{ij} \equiv 0 \pmod{p}$, $1 \leq i, j \leq 4$. In particular

$$\begin{aligned}
 & -n_1^2 n_2 \equiv d_{13} \equiv 0 \pmod{p}, \\
 (**) \quad & -n_1^2 (n_1 - n_4) \equiv d_{12} \equiv 0 \pmod{p}, \\
 & -n_3^2 n_2 \equiv d_{34} \equiv 0 \pmod{p}, \\
 & -n_3 n_4 (n_1 - n_4) \equiv d_{41} \equiv 0 \pmod{p}.
 \end{aligned}$$

The equations (**) imply either $n_1 \equiv n_3 \equiv 0 \pmod{p}$ or $n_2 \equiv n_1 - n_4 \equiv 0 \pmod{p}$.

LEMMA 7. $A(G) = AZ(G)$.

PROOF. $C_G(\langle h \rangle) = G' \cdot \langle a_2, a_4 \rangle$ if $h \in R_1$ and $C_G(\langle h \rangle) = G' \cdot \langle a_3, a_1 a_4 \rangle$ if $h \in R_2$. Let $\theta \in A(G)$. Either (i) $\theta(a_2), \theta(a_4) \in R_1$ and $\theta(a_3) \in R_2$ or (ii) $\theta(a_2), \theta(a_4) \in R_2$ and $\theta(a_3) \in R_1$.

The defining relations 2 and 3 of G imply $(\theta(a_i), h_i) = h_i^p$ for some $h_i \in R_i$, $i = 1, 2$. Let

$$\begin{aligned}
 h_1 &= a_2^r a_4^s, \\
 h_2 &= a_1^t a_3^u a_4^t, \\
 \theta(a_1) &= a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} a_4^{\alpha_4}.
 \end{aligned}$$

Therefore we get, among other equalities, the following:

$$r\alpha_1 \equiv r(\alpha_3 + 1) \equiv s\alpha_3 \equiv s(\alpha_1 - 1) \equiv (\alpha_1 - \alpha_4 - 1)u \equiv (\alpha_1 - \alpha_4 - 1)t \equiv 0 \pmod{p}.$$

The only solutions are $\alpha_1 - 1 \equiv \alpha_3 \equiv \alpha_4 \pmod{p}$.

Consider case (ii). Thus

$$\begin{aligned}
 \theta(a_1) &= a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_1-1} a_4^{\alpha_1-1}, & \theta(a_2) &= a_1^{\beta_1} a_3^{\beta_3} a_4^{\beta_1}, \\
 \theta(a_3) &= a_2^{\gamma_3} a_4^{\gamma_4}, & \theta(a_4) &= a_1^{\delta_1} a_3^{\delta_3} a_4^{\delta_1}.
 \end{aligned}$$

From the defining relations 1, 4 and 6 of G we get, among other equalities, the following:

$$\begin{aligned}
 \alpha_1 + \alpha_2 \beta_1 &\equiv \alpha_1 - \beta_1 - 1 \equiv \beta_1(\gamma_2 - 1) \equiv \beta_1(\gamma_4 - 1) \\
 &\equiv \beta_3(\gamma_4 - 1) \equiv \delta_3 \gamma_4 \equiv (\delta_3 - 1)\gamma_2 \equiv 0 \pmod{p}.
 \end{aligned}$$

Either $\beta_1 \not\equiv 0 \pmod{p}$ or $\beta_3 \not\equiv 0 \pmod{p}$. Hence $\gamma_4 - 1 \equiv \delta_3 \equiv \gamma_2 \equiv \beta_1 \equiv \alpha_1 \equiv \alpha_1 - 1 \equiv 0 \pmod{p}$, a contradiction.

Consider case (i). Thus

$$\begin{aligned}
 \theta(a_1) &= a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_1-1} a_4^{\alpha_1-1}, & \theta(a_2) &= a_2^{\beta_2} a_4^{\beta_4}, \\
 \theta(a_3) &= a_1^{\gamma_1} a_3^{\gamma_3} a_4^{\gamma_1}, & \theta(a_4) &= a_2^{\delta_2} a_4^{\delta_4}.
 \end{aligned}$$

From the defining relations 1, 4 and 6 of G we get, among other equalities, the following:

$$\begin{aligned}
 \alpha_1(\beta_2 - 1) &\equiv (\alpha_1 - 1)\beta_2 + \alpha_2 \equiv \alpha_1(\beta_4 - 1) + 1 \equiv \beta_2\gamma_1 \equiv \beta_2(\gamma_3 - 1) \\
 &\equiv \beta_4\gamma_3 \equiv \beta_4(\gamma_1 + 1) \equiv \delta_2(\gamma_1 - 1) \equiv (\delta_4 - 1)\gamma_3 \equiv 0 \\
 &\quad (\text{mod } p).
 \end{aligned}$$

If $\beta_4 \not\equiv 0 \pmod{p}$, then $\gamma_3 \equiv \gamma_1 + 1 \equiv \beta_2 \equiv \alpha_1 \equiv 0 \pmod{p}$. Thus $1 \equiv 0 \pmod{p}$, a contradiction. If $\beta_4 \equiv 0 \pmod{p}$, then $\beta_2 \not\equiv 0 \pmod{p}$ and

$$\gamma_1 \equiv \gamma_3 - 1 \equiv \delta_2 \equiv \delta_4 - 1 \equiv \alpha_1 - 1 \equiv \alpha_2 \equiv \beta_2 - 1 \equiv 0 \pmod{p}.$$

Hence $\theta \in AZ(G)$.

REFERENCES

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UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801