EXTENSIONS OF FATOU'S THEOREM TO TANGENTIAL ASYMPTOTIC VALUES¹

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ABSTRACT. Two theorems on the existence of tangential boundary values for harmonic functions on the disk are proved. One theorem is proved classically and the other is proved utilizing results concerning the maximal ideal space of H^∞ .

1. Introduction. The classical Fatou Theorem gives a condition which insures that a harmonic function on the unit disk D , $D = \{z \mid |z| < 1\}$, shall have a nontangential boundary value at a point of the boundary ∂ , $\partial = \{z \mid |z| = 1\}$. By a nontangential arc we mean the following. Let $\theta = \theta(\gamma)$ and $r = r(\gamma)$ be two continuous functions on $[0, 1]$ with $\theta(1) = \theta_0$ and $r(1) = 1$ such that $0 \leq r(\gamma) < 1$ for $0 \leq \gamma < 1$. The arc $\Gamma(\gamma)$ given by

$$\Gamma: z(\gamma) = r(\gamma)e^{i\theta(\gamma)}$$

is nontangential at $z_0 = e^{i\theta_0}$ if $(\theta(\gamma) - \theta_0)/(1 - r(\gamma))$ is bounded for $0 \leq \gamma < 1$. Γ is said to be *upper tangential* at $e^{i\theta_0}$ if

$$\lim_{\gamma \rightarrow 1^-} \frac{\theta(\gamma) - \theta_0}{1 - r(\gamma)} = \infty.$$

It is no restriction to suppose $\theta_0 = 0$ and we shall always make this assumption. Let $F(\theta) = f(e^{i\theta})$, $-\pi < \theta \leq \pi$, be an L^1 function and suppose that f is the Poisson integral of F in D ,

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2)F(t)}{1 - 2r \cos(\theta - t) + r^2} dt = (P_r * F)(\theta).$$

Then f is called the harmonic extension of F and F is the boundary function of f . Fatou's Theorem states that if

$$(1) \quad \lim_{\theta \rightarrow 0} \frac{1}{\theta} \int_0^\theta F(t) dt = \alpha$$

then $f(z) \rightarrow \alpha$ as $z \rightarrow 1$ along any nontangential arc Γ . Condition (1) is both necessary and sufficient for this to happen in the case that f is a

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bounded harmonic function, and if f is a bounded analytic function (in $H^\infty(D)$) then even the one-sided condition

$$(2) \quad \lim_{\theta \rightarrow 0^+} \frac{1}{\theta} \int_0^\theta F(t) dt = \alpha$$

is necessary and sufficient in order that f have limit α along each non-tangential arc Γ . See Loomis [2] for this converse of Fatou's Theorem, or for a simple proof in the special case when $f \in H^\infty$; see Boehme, Rosenfeld, and Weiss [1].

We shall show in §2 that whenever (1) holds $f(z)$ also tends to α along every arc Γ which is "not too tangential". The order of tangency is determined by the quantity $\theta/(1-r)$. Let

$$(3) \quad \Delta(\theta) = \sup_{0 \leq t \leq \theta} \left| \frac{1}{t} \int_0^t (F(\xi) - \alpha) d\xi \right|.$$

In §2 we show that if (1) holds (and thus $\Delta(\theta) \rightarrow 0$ as $\theta \rightarrow 0$) and Γ is any upper tangential arc such that

$$(4) \quad \theta \Delta(2\theta)/(1-r) = o(1) \quad \text{as } z \rightarrow 1, z \in \Gamma,$$

then f tends to α along Γ . In particular f tends to zero in a region which is strictly larger than any Stolz region.

In §4 we show that even in the case of H^∞ the o in our theorem cannot be changed to O .

In §3 we restrict ourselves to bounded functions and we are interested in those functions such that F is approximately continuous at $t=0$. A bounded function F is approximately continuous at $\theta=0$ with value α if and only if

$$\lim_{\theta \rightarrow 0} \frac{1}{\theta} \int_0^\theta |F(t) - \alpha| dt = 0.$$

We obtain a theorem in §3 for bounded approximately continuous functions with the $\Delta(2\theta)$ of §2 replaced by $\delta(2\theta)$ where

$$(5) \quad \delta(\theta) = \frac{1}{\theta} \int_0^\theta |F(t) - \alpha| dt.$$

Again we show that the o in

$$(6) \quad \theta \delta(2\theta)/(1-r) = o(1), \quad z \rightarrow 1, z \in \Gamma,$$

cannot be replaced by O .

The proofs in §2 are straight classical types of arguments while those in §3 use the techniques of Banach algebras. The results are

independent in that neither includes the other. The results of §3 are similar to results of Tanaka [3] and Tsuji [4] but the proofs are quite different, and our results improve their results.

2. Tangential boundary values. Let F be in $L[-\pi, \pi]$ and have period 2π , and let f be the harmonic extension to D of F . We will write $M(F; \theta)$ for the mean value of F on $[0, \theta]$,

$$M(F; \theta) = \frac{1}{\theta} \int_0^\theta F(t) dt.$$

Then the quantity Δ defined by (3) is the supremum of $|M - \alpha|$. We shall prove a supplement to Fatou's Theorem for upper tangential arcs which are not too tangential.

THEOREM 2.1. *Let F be in $L^1(-\pi, \pi)$ and let $f(z)$ be its harmonic extension to D . Suppose*

(i) *there is a constant $B > 0$ such that $|M(F; \theta)| < B$ for all $0 < |\theta| < \pi$ and*

(ii) $\lim_{\theta \rightarrow 0^+} M(F; \theta) = \alpha$.

If Γ is an upper tangential arc at 1 such that (4) holds then $f(z) \rightarrow \alpha$ as $z \rightarrow 1$ along Γ .

We need two lemmas to prove the theorem.

LEMMA 2.1. *If Γ is tangential to ∂ at 1 then $\theta P_r(\theta) \rightarrow 0$ as $z \rightarrow 1$ along Γ .*

PROOF OF LEMMA 2.1.

$$\begin{aligned} \theta P_r(\theta) &= \frac{\theta}{2\pi} \frac{1 - r^2}{1 - 2r \cos \theta + r^2} \\ &= \frac{1 + r}{2\pi} \frac{1}{\frac{1 - r}{\theta} + \frac{2r(1 - \cos \theta)}{\theta^2} \frac{\theta}{1 - r}} \\ &\sim \frac{1}{\pi} \frac{1}{\frac{1 - r}{\theta} + \frac{\theta}{1 - r}} \end{aligned}$$

as $z \rightarrow 1$, and if Γ is tangential from above to ∂ at 1 then $\theta/(1 - r) \rightarrow \infty$ as $z \rightarrow 1$ along Γ which proves the lemma.

LEMMA 2.2. *If Γ is tangential to ∂ at 1 then $\int_0^\pi P_r(t) dt \rightarrow 0$ as $z \rightarrow 1$ along Γ .*

PROOF OF LEMMA 2.2. An explicit calculation of the integral yields

$$\int_{\theta}^{\pi} P_r(t) dt = \frac{1}{\pi} \tan^{-1} \left[\frac{1-r}{(1+r) \tan(\theta/2)} \right] \sim \frac{1}{\pi} \tan^{-1} \left(\frac{1-r}{\theta} \right)$$

which is $o(1)$ as $z \rightarrow 1$ along a tangential arc Γ .

The proof of Theorem 2.1 now proceeds along the lines of the proof of Fatou's Theorem.

PROOF OF THEOREM. Integration by parts gives

$$f(z) = P_r(\theta - \pi) \int_{-\pi}^{\pi} F(t) dt + \int_{-\pi}^{\pi} P'_r(\theta - t) \left(\int_0^t F(\xi) d\xi \right) dt$$

and $P_r(\theta - \pi) \rightarrow 0$ as $z \rightarrow 1$ (along any arc). Thus

$$\begin{aligned} f(z) &= \left[\int_{-\pi}^0 + \int_0^{2\theta} + \int_{2\theta}^{\pi} \right] P'_r(\theta - t) \left(\int_0^t F(\xi) d\xi \right) dt + o(1) \\ &= I_1(r, \theta) + I_2(r, \theta) + I_3(r, \theta) + o(1) \end{aligned}$$

as $z \rightarrow 1$ along Γ , where I_1 , I_2 , and I_3 are the first, second, and third integrals, respectively, in the above equation. We shall first show that I_1 and I_3 tend to zero along *any* arc which is tangential to ∂ at 1 when (i) holds, and then show that I_2 tends to α along the arcs prescribed by the theorem when (ii) holds.

We can without loss of generality suppose $\alpha = 0$.

Because of assumption (i) in the theorem we have

$$|I_1(r, \theta)| \leq B \int_{-\pi}^0 |t P'_r(\theta - t)| dt.$$

Changing variables in this integral gives

$$\begin{aligned} |I_1(r, \theta)| &\leq B \left(\int_{\theta}^{\pi} (t - \theta) |P'_r(t)| dt + \int_{\pi}^{\pi+\theta} (t - \theta) |P'_r(t)| dt \right) \\ &\leq B \int_{\theta}^{\pi} (\theta - t) P'_r(t) dt + o(1) \quad \text{as } z \rightarrow 1, z \in \Gamma, \end{aligned}$$

and integrating by parts gives

$$|I_1(r, \theta)| \leq B \left[\frac{1}{2\pi} \frac{1-r}{1+r} (\theta - \pi) + \int_{\theta}^{\pi} P_r(t) dt \right] + o(1).$$

Lemma 2.2 shows that $I_1(r, \theta) = o(1)$ as $z \rightarrow 1$ along any tangential arc.

The proof that $I_3(r, \theta) = o(1)$ as $z \rightarrow 1$ along a tangential arc is nearly identical to the above proof and will not be repeated.

It only remains to show that if Γ is not too tangential then $I_2(r, \theta) = o(1)$ as $z \rightarrow 1$ along Γ . Now

$$\begin{aligned}
|I_2(r, \theta)| &= \left| \int_0^{2\theta} t P_r'(\theta - t) M(F; t) dt \right| \\
&\leq \Delta(2\theta) \int_{-\theta}^{\theta} |(\theta - t) P_r'(t)| dt \\
&\leq \Delta(2\theta) \left[-2\theta \int_0^{\theta} P_r'(t) dt \right] \\
(7) \quad &= \Delta(2\theta) \left[2\theta(P_r(0) - P_r(\theta)) \right] \\
(8) \quad &= 2\theta \Delta(2\theta) P_r(0) + o(1), \quad z \rightarrow 1, z \in \Gamma \\
&= \frac{\theta \Delta(2\theta)(1+r)}{\pi(1-r)} + o(1), \quad z \rightarrow 1, z \in \Gamma
\end{aligned}$$

which proves the theorem. Here, in going from (7) to (8) we have used Lemma 2.1 and the fact that $\Delta(2\theta) \rightarrow 0$ along Γ .

The following two cases are of special interest. In each case condition (i) of the theorem is automatically satisfied.

COROLLARY 2.1. *If $F \in L(-\pi, \pi)$ and (1) is satisfied then $f(z) \rightarrow \alpha$ along any upper tangential arc such that*

$$\theta \Delta(2\theta)/(1-r) = o(1) \quad \text{as } z \rightarrow 1, z \in \Gamma.$$

COROLLARY 2.2. *If $F \in L^\infty(-\pi, \pi)$ and (2) is satisfied then $f(z) \rightarrow \alpha$ along any upper tangential arc Γ such that*

$$\theta \Delta(2\theta)/(1-r) = o(1) \quad \text{as } z \rightarrow 1, z \in \Gamma.$$

3. Approximately continuous boundary functions for bounded harmonic functions. The theorem of this section is a variant of Theorem 2.1 in the case that the boundary function is a bounded function and is approximately continuous at the point $z=1$. The variation is obtained by replacing the Δ of Theorem 2.1 by the δ of equation (5). The interesting point here is that we get rid of the Sup which is involved in Δ .

THEOREM 3.1. *Let F be in $L^\infty(-\pi, \pi)$ and $f(z)$ be its harmonic extension to D . Suppose F is approximately continuous from above at 0 with value α , i.e. $\delta(\theta) \rightarrow 0$ as $\theta \rightarrow 0^+$. Then if Γ is upper tangential to ∂ at 1 and is such that (6) holds we have $f(z) \rightarrow \alpha$ as $z \rightarrow 1$ along Γ .*

REMARKS. In §4 we will point out how neither of the two theorems, Theorem 2.1 and Theorem 3.1, includes the other and that they each are in a sense best possible. For now it will be well to mention the relationship of these theorems to results of Tsuji and Tanaka. The

theorem of Tsuji [3, Theorem I], is valid for L^1 functions. Tsuji computes the symmetric mean value,

$$\epsilon(\theta) = (1/2\theta) \int_{-\theta}^{\theta} |F(t) - \alpha| dt,$$

and has the conclusion that $f(re^{i\theta}) \rightarrow \alpha$ in the domain bounded by the curve $1-r = |\theta| \sqrt{(\epsilon(2|\theta|))}$. Tanaka's result [4, Theorem 4] is proved for H^∞ functions and (although he does not state it explicitly) has the conclusion that $f(re^{i\theta}) \rightarrow \alpha$ for all nontangential approaches and also for approaches from above which are no more tangential than $1-r = \theta \sqrt{(\delta(2\theta))}$, where δ is as in Theorem 3.1. We will prove that as was the case in §2 with Theorem 2.1 the o condition in Theorem 3.1 cannot be replaced by a corresponding O condition.

The proof of Theorem 3.1 begins with an estimate for certain harmonic measures. Given a measurable set $S \subset \partial$ let $a = a(S, \theta) = \lambda(S \cap [0, 2\theta])$ where λ is Lebesgue measure on ∂ . Let

$$u_S(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_S(e^{it}) \frac{1-r^2}{1-2r \cos(\theta-t) + r^2} dt$$

be the harmonic measure of S at $z = re^{i\theta} \in D$.

LEMMA 3.2. *If S is a measurable subset of $\{e^{i\theta} | 0 \leq \theta \leq \pi\}$ and u_S is its harmonic measure, then for $re^{i\theta} \in D$ and θ close enough to zero*

$$u_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1} \left\{ \frac{a/2\theta}{\frac{1-r}{\tan(\theta/2)} + 4 \frac{\theta}{1-r} [1 - (a/2\theta)]} \right\}$$

where $a = a(S, \theta)$ is as above.

PROOF. Because S is a subset of the upper portion of ∂ and the integrand involved is positive, we may write

$$\begin{aligned} u_S(re^{i\theta}) &= \int_0^{\pi} \chi_S(e^{it}) P_r(\theta-t) dt \\ &\geq \int_0^{2\theta} \chi_S(e^{it}) P_r(\theta-t) dt \\ &= \int_{-\theta}^{\theta} \chi_S(e^{i(\theta-t)}) P_r(t) dt. \end{aligned}$$

Because of the shape of the Poisson kernel we may continue the estimate by replacing S by the two intervals $[-\theta, -\theta + (a/2)] \cup [\theta - (a/2), \theta]$. Then, since P_r is even

$$\begin{aligned}
 u_S(re^{i\theta}) &\geq 2 \int_{\theta-(a/2)}^{\theta} P_r(\theta-t) dt \\
 &= \frac{2}{\pi} \tan^{-1} \left\{ \frac{1+r}{1-r} \frac{\tan(\theta/2) - \tan((2\theta-a)/4)}{1 + \left(\frac{1+r}{1-r}\right)^2 \tan(\theta/2) \tan((2\theta-a)/4)} \right\}
 \end{aligned}$$

where the last step is an explicit integration. In the last expression above we factor out $\tan(\theta/2)$ from the numerator and denominator of the fraction in the argument of $\tan^{-1}$. Then, because for small θ we have $0 \leq (2\theta-a)/4 \leq \theta/2 < \pi/2$, we also have $1 - [\tan(2\theta-a)/4] \cdot [\tan(\theta/2)]^{-1} \geq a/2\theta$. Furthermore, for small enough θ , $\tan(2\theta-a)/4 \leq (2\theta-a)/2$. Using these estimates we obtain

$$u_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1} \left\{ \frac{1+r}{1-r} \frac{a/2\theta}{[\tan(\theta/2)]^{-1} + \left(\frac{1+r}{1-r}\right)^2 \frac{2\theta-a}{2}} \right\}.$$

Performing some simple algebraic manipulations and using the fact that $0 < r < 1$ we obtain the estimate claimed.

PROOF OF THEOREM 3.1. Let $\epsilon > 0$, let $S_\epsilon = \{t \mid |f(t) - \alpha| < \epsilon\}$, and let $a(\epsilon) = \lambda(S_\epsilon \cap [0, 2\theta])$. We compute

$$\delta(2\theta) \geq \frac{1}{2\theta} \int_{[0, 2\theta] - S_\epsilon} |F(t) - \alpha| dt \geq \epsilon \frac{2\theta - a(\epsilon)}{2\theta}.$$

Then, from Lemma 3.2

$$u_{S_\epsilon}(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1} \left\{ \frac{a/2\theta}{\frac{1-r}{\tan(\theta/2)} + 4 \frac{\theta\delta(2\theta)}{1-r} \frac{1}{\epsilon}} \right\}.$$

Consequently, if $re^{i\theta}$ tends to 1 tangentially so that $(1-r)/\tan(\theta/2) \rightarrow 0$ and, in addition, if $\theta\delta(2\theta)/(1-r) \rightarrow 0$ we have for each $\epsilon > 0$,

$$\lim u_{S_\epsilon}(re^{i\theta}) = 1.$$

At this point we use the principle (see [1, Theorem 2.2]) that for any homomorphism, h , lying in the fiber, $\mathfrak{D}_1$, above 1 in the maximal ideal space of H^∞ , $\mu_h(\tilde{S}_\epsilon) = u_{S_\epsilon}(h)$ ($= h(u_{S_\epsilon})$); where μ_h is the representing measure for h , and $\tilde{S}_\epsilon$ consists of those homomorphisms, ϕ , in the part of the Šilov boundary of H^∞ in $\mathfrak{D}_1$ for which $\phi(\chi_{S_\epsilon}) = 1$. Thinking of f as the extension of F to the maximal ideal space of H^∞ we recall that the range of f on $\tilde{S}_\epsilon$ consists of the essential cluster values of F

at 1 through S_ϵ . Thus, on $\tilde{S}_\epsilon$, $|f - \alpha| \leq \epsilon$ and so for any h such that $\mu_h(\tilde{S}_\epsilon) = 1$ we have

$$|f(h) - \alpha| \leq \int_{\tilde{S}_\epsilon} |f - \alpha| d\mu_h \leq \epsilon \mu_h(\tilde{S}_\epsilon) = \epsilon.$$

Therefore, if h is any homomorphism which is approached tangentially from above but less tangentially than any of the curves $\theta\delta(2\theta)/(1-r) = k \in (0, \infty)$, we have from our original estimates that $\mu_h(\tilde{S}_\epsilon) = \mu_{\tilde{S}_\epsilon}(h) = 1$; and so, for any $\epsilon > 0$, $|f(h) - \alpha| \leq \epsilon$ or $f(h) = \alpha$. This was our assertion.

4. Conclusions. (A) First we shall give an example to show that the $o(1)$ in Theorem 2.1 cannot be replaced by $O(1)$ even if f is in H^∞ . Let $F(t) = \exp[-i \operatorname{ctn}(t/2)]$. Then

$$f(z) = (F * P_r)(\theta) = \exp((z+1)/(z-1))$$

is a bounded analytic function in D . It is not difficult to show that for this F ,

$$M(F; \theta) = O(\theta) \quad \text{as } \theta \rightarrow 0$$

so that with $\alpha = 0$ we have

$$\Delta(\theta) = O(\theta) \quad \text{as } \theta \rightarrow 0.$$

Thus if Γ is such that

$$\theta/(1-r) = o(1/\theta) \quad \text{as } z \rightarrow 1, z \in \Gamma,$$

then $f(z) \rightarrow 0$ along Γ . This is equivalent to saying that Γ is eventually inside every circle $|z - (1-\rho)| = \rho$, $0 < \rho < 1$, since for these circles

$$(9) \quad \frac{\theta}{1-r} \sim \frac{2\rho}{1-\rho} \frac{1}{\theta} = O\left(\frac{1}{\theta}\right) \quad \text{as } z \rightarrow 1, |z - (1-\rho)| = \rho.$$

Thus our Theorem 2.1 says that $f(z) \rightarrow 0$ along any arc which is less tangent than every circle $|z - (1-\rho)| = \rho$, $0 < \rho < 1$. This is the best possible result for this f since f has constant absolute value on each such circle. In fact

$$(10) \quad |f(z)| = \exp(1 - 1/\rho), \quad 0 < \rho < 1, |z - (1-\rho)| = \rho,$$

and thus if $\theta/(1-r) \neq o(1/\theta)$ as $z \rightarrow 1$ along Γ , by (9) Γ is frequently outside some circle $|z - (1-\rho)| = \rho$, and, by (10), $f(z)$ does not tend to zero along Γ . It follows that for this $f \in H^\infty$ we have $f(z) \rightarrow 0$ along Γ if and only if

$$\theta\Delta(2\theta)/(1-r) = o(1) \quad \text{as } z \rightarrow 1, z \in \Gamma.$$

(B) Theorem 3.1 does not apply to the above example since the given boundary function is not approximately continuous at $t=0$. We shall now give an example to show that Theorem 3.1 also cannot be improved by using $O(1)$ in place of $o(1)$.

Pick positive values θ_n and l_n decreasing to zero such that

$$\theta_{n+1} < \frac{\theta_n}{4}, \quad l_n = o(\theta_n), \quad \sum_{k=1}^{\infty} l_k = o(l_n), \quad n = 1, 2, \dots,$$

and let $E = \bigcup_n \{e^{it} \mid \theta_n - l_n < t < \theta_n + l_n\}$. Then if $F = \chi_E$ is the boundary function we have

$$\delta(2\theta_n) = \frac{2l_n + o(l_n)}{2\theta_n} = \frac{l_n}{\theta_n} + o\left(\frac{l_n}{\theta_n}\right) = o(1).$$

Thus F is approximately continuous at $z=1$ with value 0. If we take $1-r_n = l_n$ then

$$\theta_n/(1-r_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

and

$$\theta_n \delta(2\theta_n)/(1-r_n) = 1 + o(1) = O(1) \quad \text{as } n \rightarrow \infty.$$

But the harmonic extension of F is f where

$$\begin{aligned} f(r_n e^{i\theta_n}) &> \int_{\theta_n - l_n}^{\theta_n + l_n} P_{r_n}(\theta_n - t) dt = 2 \int_0^{l_n} P_{r_n}(t) dt \\ &= 1 - \frac{2}{\pi} \tan^{-1} \left[\frac{1-r_n}{(1+r_n) \tan(l_n/2)} \right] \sim \frac{1}{2}. \end{aligned}$$

Thus f does not tend to zero along an arc through the points $z_n = r_n e^{i\theta_n}$.

(C) It should be noted that the counterexamples in (A) and (B) above are of different type in that in (A) we have shown that along no curve which is so tangent that $\theta\Delta(2\theta)/(1-r) \neq o(1)$ does $f(z) \rightarrow \alpha$. While in (B) we have only shown that, for some curve such that $\theta\delta(2\theta)/(1-r) = O(1)$, $f(z) \rightarrow \alpha$ fails.

(D) Neither of the Theorems 2.1 and 3.1 include the other even in the case of bounded functions. The example in (A) above shows that 3.1 does not include 2.1. An example to show the converse is somewhat more complicated to construct but can be constructed.

If the Δ in Theorem 2.1 or δ in Theorem 3.1 could be replaced by $M(F, 2\theta)$ then both theorems would result from that one theorem, but we have not been able to prove this result or to find a counterexample and we must leave it as an open question.

BIBLIOGRAPHY

1. T. Boehme, M. Rosenfeld and M. L. Weiss, *Relations between bounded analytic functions and their boundary functions*, J. London Math. Soc. (2) **1** (1969), 609–618.
2. L. Loomis, *The converse of the Fatou theorem for positive harmonic functions*, Trans. Amer. Math. Soc. **53** (1943), 239–250. MR **4**, 199.
3. C. Tanaka, *On the metric cluster values of the bounded regular function in the unit disk*, Mem. School Sci. Engrg. Waseda Univ. Tokyo No. 31 (1967), 119–129.
4. M. Tsuji, *On Fatou's theorems on Poisson integrals*, Japan J. Math. **15** (1938), 13–18.

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