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A CLASS OF ERGODIC TRANSFORMATIONS HAVING SIMPLE SPECTRUM

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Abstract. A class of ergodic, measure-preserving, invertible point transformations is defined, called class S. Any measure-preserving point transformation induces a unitary operator on the Hilbert space of $\mathcal{L}_2$-functions. A theorem is proved here which implies that the operator induced by any transformation in class S has simple spectrum. [It is then a known result that the transformations in class S have zero entropy.]

Let $(X, \mathfrak{F}, \mu)$ be a measure space, isomorphic to the unit interval with Lebesgue measure. A measurable, measure-preserving, invertible point transformation of X is called an automorphism of $(X, \mathfrak{F}, \mu)$. A class of automorphisms, called class S for brevity, is defined below (Definition (4)). The purpose of this paper is to prove the following theorem:

(1) Theorem. *Let τ be an automorphism in class S. Then there exist arbitrarily small sets whose characteristic functions each generate $\mathcal{L}^2(d\mu)$ under the action of the unitary operator U_τ, where U_τ is defined by $U_\tau f(\tau x)=f(x)$. In particular U_τ has simple spectrum.*

(2) Definition. Let H be a Hilbert space, T a bounded normal operator on H. Let $v\in H$. Let $H(v)$ consist of the closure of the set of all elements of the form $P(T, T^*)v$, where $P(T, T^*)$ denotes a polynomial in T and T^*. To say that a vector $v\in H$ "generates H under the action of T" means that $H=H(v)$.

(3) Definition. Let $\xi=\{A_i \mid 1\leqq i\leqq m\}$ be a finite, ordered collection of mutually disjoint measurable sets. Then ξ is called a *partition*. The union of the members of ξ need not be the whole space. Let ξ_k be a sequence of partitions with the property that for every measurable set E, there exists a sequence of sets E_k such that each E_k is a union of members of ξ_k, and $\mu(E \bigtriangleup E_k)\to 0$ as $k\to\infty$. Then it will be said that $\xi_k\to\epsilon$. Here ϵ denotes the partition of the whole space into one-point sets.

(4) Definition. Let τ be an automorphism of $(X, \mathfrak{F}, \mu)$, $\xi=\{A_i \mid 1\leqq i\leqq m\}$ a partition. If $\tau A_i=A_{i+1}$ for $1\leqq i\leqq m-1$, ξ will

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be called a τ-partition. If a sequence of τ-partitions ξ_k exists such that $\xi_k \to \epsilon$, then τ will be said to be in class S.

The automorphisms constructed in [2] and [5] are examples of class S automorphisms. Some general properties of automorphisms in class S are given in [1] and [3].

The first part of the proof of Theorem (1) is completed in Lemma (9), below. Some simple definitions and lemmas are given first, to simplify the proof of Lemma (9).

(5) DEFINITION. Let $\xi = \{A_i \mid 1 \leqq i \leqq m\}$, $\eta = \{B_j \mid 1 \leqq j \leqq n\}$ be partitions. $\eta \leqq \xi$ will be written to mean that each member of η is a union of members of ξ. If E is a set, $E \leqq \xi$ will mean $\{E\} \leqq \xi$. If $m = n$ let $\rho(\xi, \eta) = \sum_{i=1}^{m} \mu(A_i \triangle B_i)$. For any ξ and η, let $D(\eta, \xi)$ be the smallest number of the form $\rho(\eta, \eta')$, where $\eta' \leqq \xi$. Let $D(E, \xi) = D(\{E\}, \xi)$.

Thus $\xi_k \to \epsilon$ means $D(E, \xi_k) \to 0$ as $k \to \infty$ for all measurable sets E.

(6) DEFINITION. Let $\xi = \{A_i \mid 1 \leqq i \leqq m\}$ be a τ-partition. Let $\eta = \{B_j \mid 1 \leqq j \leqq n\}$ be a partition. Let $\tau - D(\eta, \xi)$ be the smallest number of the form $\rho(\eta, \eta')$, where $\eta' = \{B'_j \mid 1 \leqq j \leqq n\}$ is a τ-partition, $\eta' \leqq \xi$, and also: if I'_j denotes the set of indices i such that $A_i \subset B'_j$, $1 \leqq j \leqq n$, then $I'_{j+1} = \{i + 1 \mid i \in I'_j\}$, $1 \leqq j \leqq n-1$.

(7) LEMMA. *Let ξ and ξ' be τ-partitions with the same number of elements. Let η be a partition. Then*

$$\tau - D(\eta, \xi') \leqq \tau - D(\eta, \xi) + \rho(\xi, \xi').$$

PROOF. Follows directly from the definitions.

(8) LEMMA. *Let $\xi = \{A_i \mid 1 \leqq i \leqq m\}$ and $\eta = \{B_j \mid 1 \leqq j \leqq n\}$ be τ-partitions. Then*

$$\tau - D(\eta, \xi) \leqq nD(B_1, \xi) + n\mu(A_1).$$

PROOF. Let F be a measurable set, $F \leqq \xi$, such that $\mu(B_1 \triangle F) = D(B_1, \xi)$.

Let G be the set obtained by, first, removing from F all sets A_i, such that $\mu(B_1 \cap A_i) \leqq \frac{1}{2}\mu(A_i)$, and, second, removing that A_i of largest index i still remaining in F.

Let I'_1 be the set of indices i such that $A_i \subset G$. Because η is a τ-partition, it follows that any two indices in I'_1 must differ by at least n, and also that the largest index in I'_1 is less than $m - n + 1$.

Let $B'_j = \tau^{j-1}G$, $1 \leqq j \leqq n$. Define the τ-partition $\eta' = \{B'_j \mid 1 \leqq j \leqq n\}$. It follows from the facts about I'_1 just stated, that $\eta' \leqq \xi$, and also that the conditions on η' given in Definition (6) are satisfied. Hence by definition $\tau - D(\eta, \xi) \leqq \rho(\eta, \eta')$.

Furthermore, it is clear from the construction of G that $\mu(B_1 \triangle G) \leqq D(B_1, \xi)+\mu(A_i)$. Since $\rho(\eta, \eta')=n\mu(B_1 \triangle G)$ the lemma is proved.

(9) LEMMA. *Let ξ_k be a sequence of τ-partitions, $\xi_k \to \epsilon$. Then there exists a sequence η_k of τ-partitions, $\eta_k \to \epsilon$, such that $\eta_k \leqq \eta_{k+1}$, for all k.*

PROOF. By (8), it can be assumed that $\tau-D(\xi_k, \xi_{k+1})<\delta_k$, $k \geqq 1$, where $\sum_{k=1}^{\infty} \delta_k<\infty$.

A doubly infinite sequence of τ-partitions ξ_k^r, $r \geqq 0$, $k \geqq 1$, will now be defined by induction.

Let $\xi_k^0=\xi_k$, $k \geqq 1$.

Having defined ξ_k^r, $k \geqq 1$, let ξ_k^{r+1} be some τ-partition such that

$$\tau - D(\xi_k^{r+1}, \xi_{k+1}^r) = 0, \tag{10}$$

and

$$\rho(\xi_k^r, \xi_k^{r+1}) = \tau - D(\xi_k^r, \xi_{k+1}^r). \tag{11}$$

Suppose that for some $r \geqq 0$, and all $k \geqq 1$,

$$\rho(\xi_k^r, \xi_k^{r+1}) < \delta_{r+k}. \tag{12}$$

Then since

$$\rho(\xi_k^{r+1}, \xi_k^{r+2}) = \tau - D(\xi_k^{r+1}, \xi_{k+1}^{r+1}),\ \tau - D(\xi_k^{r+1}, \xi_{k+1}^r) = 0,$$

$$\therefore\ \rho(\xi_k^{r+1}, \xi_k^{r+2}) \leqq \rho(\xi_{k+1}^r, \xi_{k+1}^{r+1}) < \delta_{r+1+k},$$

using Lemma (7) and the assumption.

Hence (12) holds with r replaced by $r+1$. Since (12) holds for $r=0$ and all k, it is true for all r, k, by induction.

From (12),

$$\rho(\xi_k^r, \xi_k^{r+p}) < \sum_{q=0}^{p-1} \delta_{q+r+k}, \tag{13}$$

for $r \geqq 0$, $p \geqq 0$, $k \geqq 1$.

Hence ξ_k^r is a Cauchy sequence in r for each k, and it follows that there exists a partition η_k such that $\rho(\xi_k^r, \eta_k) \to 0$ as $r \to \infty$, for each $k \geqq 1$. It is clear that η_k must be a τ-partition.

Taking limits in (10) shows that $\eta_k \leqq \eta_{k+1}$, $k \geqq 1$.

Finally, since $\xi_k \to \epsilon$ and $\rho(\xi_k, \eta_k) \leqq \sum_{l=k}^{\infty} \delta_l \to 0$ as $k \to \infty$, it follows that $\eta_k \to \epsilon$. This completes the proof of Lemma (9).

The next lemma concerns an abstract Hilbert space H, and a bounded normal operator T. For any subspace V of H, let E_V denote the orthogonal projection on V. Suppose a sequence of elements v_n exists, such that $v_n \to 0$ and $E_{H(v_n)} \to I$ (strongly).

(14) LEMMA. *There exists a subsequence v_{n_k} such that $v=\sum_k v_{n_k}\in H$ and $H(v)=H$.*

PROOF. Let S be any bounded operator that commutes with T and T^*. Then, for each n, $E_{H(v_n)}\ SE_{H(v_n)}$ commutes with T and T^* on $H(v_n)$. Since T has simple spectrum as an operator on each $H(v_n)$, it can be shown using the spectral theorem, or rather, using the simplest case of the spectral representation theorem, that there exists a sequence P_m^n of polynomials in T and T^* such that: $P_m^n \to E_{H(v_n)}\ SE_{H(v_n)}$ on $H(v_n)$ as $m\to\infty$, and $\|P_m^n\|\leqq\|S\|$, for all n, m.

H is clearly separable. Let $\{u_1, u_2, \cdots\}$ be a basis for H.

Since $E_{H(v_n)}\to I$, it is easy to see that sequences n_k, m_k can be chosen, such that

$$\|(P_{m_k}^{n_k}-S)u_l\| < 1/k, \quad 1\leqq l\leqq k.$$

Hence $P_{m_k}^{n_k}\to S$ as $k\to\infty$.

Thus any bounded operator S which commutes with T and T^* is a strong limit of polynomials in T and T^*. Since T has this property it follows at once from the spectral representation theorem that T has simple spectrum.

Hence it may be assumed without loss of generality that $H=\mathfrak{L}_2(Y, \mathcal{G}, m)$, where m is a finite Borel measure on a compact subset Y of the complex plane, and T is defined by $Tf=zf$. The elements v_n will be written as f_n. The properties of the f_n can now be expressed by: $\|f_n\|_2\to 0$ as $n\to\infty$, and $m(\{z|f_n(z)=0\})\to 0$ as $n\to\infty$. To prove the lemma it is sufficient to show that a subsequence f_{n_k} exists such that $f=\sum_k f_{n_k}\in\mathfrak{L}_2(dm)$ and $|f|>0$ a.e. (dm). This is a straightforward step, so the lemma is proved.

(15) PROOF OF THEOREM (1). By Lemma (9), a sequence ξ_k of τ-partitions exists, such that $\xi_k\to\epsilon$ and $\xi_k\leqq\xi_{k+1}$ for all k. Let $\xi_k=\{A_i(k)\,|\,1\leqq i\leqq m(k)\}$.

It is easy to see that a disjoint sequence of sets $E_n=A_{i_n}(k_n)$ can always be found.

Let $v_n=\chi_{E_n}\in\mathfrak{L}_2(d\mu)$, let $T=U_\tau$.

Since $\xi_k\to\epsilon$, $\therefore E_{H(v_n)}\to I$ strongly. Clearly $v_n\to 0$.

By Lemma (14), there exists a subsequence v_{n_k} such that

$$v=\sum v_{n_k}\in\mathfrak{L}_2(d\mu) \quad\text{and}\quad H(v)=\mathfrak{L}_2(d\mu).$$

But $v=\chi_E$, where $E=\bigcup_k E_{n_k}$.

Clearly E can be made as small as desired, so the theorem is proved.

It is known (cf. [6, §14.4]), that any automorphism with simple spectrum has zero entropy.

It may also be noted that Lemma (9) implies that the class S defined here is precisely the class of those automorphisms which may be constructed by the stacking method, using single columns (cf. [**4**, §6]).

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