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A NEW CHARACTERIZATION OF DEDEKIND DOMAINS

E. W. JOHNSON AND J. P. LEDIAEV

ABSTRACT. In this note it is shown that a Noetherian ring R is a Dedekind domain if every maximal ideal M of R satisfies the cancellation law: if A and B are nonzero ideals of R and $MA=MB$, then $A=B$.

Let R be a Noetherian domain (commutative with 1). And let S be the semigroup of ideals of R under multiplication. It is well known that R is a Dedekind Domain if, and only if, every element $A\in S$ satisfies the cancellation law: if B, $C\in S$ and $A\neq 0$, then $AB=AC$ implies $B=C$. Since a Dedekind domain has the property that every ideal is a product of primes, however, it is natural to ask if the assumption that every ideal is cancellable is necessary. In this note we show that a Noetherian ring is a Dedekind domain if every maximal ideal is cancellable.

For an extensive bibliography on Dedekind domains we refer the reader to [1].

The main tool used in the following is the theorem, due to Samuel [2], that if Q is an ideal primary for the maximal ideal of a local ring R, then for sufficiently large values of n, the length of R/Q^n is a polynomial in n of degree equal to the rank of M. We denote this polynomial by $p_Q(x)$.

We begin with the following:

LEMMA. *Let R be a local ring in which the maximal ideal M satisfies the cancellation law. Then either $M=0$ or M has rank* 1.

PROOF. Since M satisfies the cancellation law, either $M=0$ or $0:M=0$. In the second case, set $M=(a_1, \cdots, a_d)$ and let $p(x)$ be the polynomial $p_M(x+1)-p_M(x)$. Then for sufficiently large values of n, $p(n)$ is the length of the R-module M^n/M^{n+1}, which is also the number of elements in a minimal base for M^n. Now, for all $n\geqq 1$, $M^{nd+n}=M^{nd}(a_1^n, \cdots, a_d^n)$, so, by cancellation, $M^n=(a_1^n, \cdots, a_d^n)$. Hence $p(n)\leqq d$ for all sufficiently large n. Since $0:M=0$, it follows that $p(x)$ has degree 0, and therefore that $p_M(x)$ has degree 1. Hence M has rank 1. Q.E.D.

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THEOREM. *Let R be a Noetherian ring such that every maximal ideal satisfies the cancellation law. Then R is a Dedekind Domain.*

PROOF. Assume that R is not a field. It suffices to show that for every maximal ideal M, R_M is a regular local ring of altitude 1. To do this, fix M and set $\bar{R}=R_M$. We adopt the notation that for any ideal A of R, $\bar{A}=AR_M$. Then $\bar{A}\bar{M}:\bar{M}=(AM:M)R_M=\bar{A}$, so the maximal ideal $\bar{M}$ of the local ring $\bar{R}$ is cancellable. Since $\bar{M}\neq 0$, $\bar{M}$ has rank 1 by the Lemma. Clearly, $\bar{M}$ is not a prime of 0 in $\bar{R}$, so there exists an element $a\in\bar{R}$ such that $a\in\bar{M}$, $a\notin\bar{M}^2$, and a is not an element of any prime of 0 (see, for example, [**3**, p. 406]). Then the ideal (a) is primary for $\bar{M}$, so there exists an integer k such that $\bar{M}^k\not\subseteq(a)$ and $\bar{M}^{k+1}\subseteq(a)$ (where $\bar{M}^k=\bar{R}$ if $k=0$). Hence $\bar{M}^{k+1}=\bar{M}^{k+1}\cap(a)=(\bar{M}^{k+1}:(a))(a)$; and therefore either $\bar{M}^{k+1}\subseteq\bar{M}(a)$ or $\bar{M}^{k+1}=(a)$. However, if $\bar{M}^{k+1}\subseteq\bar{M}(a)$, then $\bar{M}(a)=\bar{M}(\bar{M}^k+(a))$ and $(a)=\bar{M}^k+(a)$, which contradicts the choice of k. Hence $\bar{M}^{k+1}=(a)$, so by the choice of a, $k=0$ and $\bar{M}$ is principal. Since $\bar{M}$ is not a prime of 0 in $\bar{R}$, this completes the proof.

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UNIVERSITY OF IOWA, IOWA CITY, IOWA 52240