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MINIMAL NONNILPOTENT SOLVABLE LIE ALGEBRAS

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ABSTRACT. We shall say that a solvable Lie algebra L is a minimal nonnilpotent Lie algebra if L is nonnilpotent but all proper subalgebras of L are nilpotent. It is shown here that if L is a minimal nonnilpotent Lie algebra, then L is the vector space direct sum of A and F where A is an ideal in L, F is a one-dimensional subalgebra of L, either A is a minimal ideal of L or the center of A coincides with the derived algebra, A', of A and in either case F acts irreducibly on A/A'.

P. Hall and G. Higman have shown in [5] that a nonnilpotent finite group G all of whose proper subgroups are nilpotent can be considered as the product of subgroups P and Q where P is cyclic of prime power order, p^α, Q is an invariant q-subgroup of G, $q \neq p$, $\phi(P) \leqq Z(G)$ and Q is either elementary abelian or $\phi(Q) = Z(Q) = [Q, Q]$ where in either case P acts irreducibly on $Q/\phi(Q)$. We shall find a result on solvable Lie algebras which is roughly analogous to this result.

Let L be a finite-dimensional solvable Lie algebra and let M be a self-normalizing maximal subalgebra of L. The maximal ideal of L contained in M will be called the core of M. The intersection of all maximal subalgebras of L will be denoted by $\phi(L)$ and is an ideal in L by Lemma 3.4 of [2]. The derived algebra of L will be denoted by L' and the center of L by $Z(L)$. We first show the following result, the group theory analogue of which is shown in [4].

PROPOSITION. *Let L be a solvable Lie algebra and let H be a self-normalizing maximal subalgebra of L. Let K be the core of H. Then*

(1) *L/K contains a unique minimal ideal A/K,*
(2) *L/K is the semidirect sum of A/K and H/K,*
(3) *$\phi(L/K) = 0$, and*
(4) *L/K is not nilpotent.*

PROOF. Let $\overline{H} = H/K$, $\overline{L} = L/K$ and $\overline{A}$ be a minimal ideal of $\overline{L}$. Since $\overline{A}$ is abelian and $\overline{H}$ is maximal in $\overline{L}$, $\overline{A} \cap \overline{H}$ is an ideal in $\overline{L}$ contained in $\overline{H}$. Since $\overline{H}$ contains no nonzero ideals of $\overline{L}$, $\overline{A} \cap \overline{H} = 0$. Then, since $\overline{H}$ is maximal in $\overline{L}$, $\overline{A} + \overline{H} = \overline{L}$. If $\overline{B}$ is another minimal ideal of $\overline{L}$, then the centralizer of $\overline{A}$ in $\overline{L}$, $C_{\overline{L}}(\overline{A})$ properly contains $\overline{A}$. Then $\overline{H} \cap C_{\overline{L}}(\overline{A})$ is an ideal in $\overline{L}$ contained in $\overline{H}$, hence $\overline{H} \cap C_{\overline{L}}(\overline{A}) = 0$ and,

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therefore, $\bar{A}$ is unique. Since $\bar{H}$ is maximal in $\bar{L}$, $\phi(\bar{L})\subseteq\bar{H}$. But $\phi(\bar{L})$ is an ideal in $\bar{L}$, hence $\phi(\bar{L})=0$. Finally, since $\bar{H}$ is a self-normalizing maximal subalgebra of $\bar{L}$, $\bar{L}$ is not nilpotent by Proposition 3, p. 56 of [3].

THEOREM 1. *Let L be a nonnilpotent, solvable Lie algebra all of whose proper subalgebras are nilpotent. Then L is the vector space direct sum of A and F where A is an ideal of L, F is a one-dimensional subalgebra of L and either A is a minimal ideal of L or $A'=Z(A)$. In either case, F acts irreducibly on A/A'.*

PROOF. By the theorem of [1], L contains a self-normalizing maximal subalgebra H and H is clearly a Cartan subalgebra of L. Let K be the core of H in L. By the Proposition, L/K contains a unique minimal ideal A/K which complements H/K in L/K. Then $L/A \simeq H/K$ and, since H is nilpotent, L/A is nilpotent. Since L/K is not nilpotent and A/K is abelian, using Engel's Theorem, there exists $x\in L/K$, $x\notin A/K$ such that ad x is not nilpotent. Since H/K is nilpotent and H/K complements A/K in L/K, ad x restricted to A/K is not nilpotent. Then the subalgebra B/K of L/K generated by A/K and x is not nilpotent and since A/K is an ideal of L/K, $\dim B/K=1+\dim A/K$. Hence $B/K=L/K$ and $\dim H/K=\dim L/A=1$. Hence there exists a one-dimensional subalgebra F of L which is a complement of K in H and is also a complement of A in L.

Let $L=H+L_1$ be the Fitting decomposition of L with respect to H. Then H/K is a Cartan subalgebra of L/K and L_1+K/K is the Fitting one-component of L/K with respect to H/K. Since $L/K=A/K+H/K$, A/K is a minimal ideal of L/K and L/K is not nilpotent, H/K acts nontrivially and irreducibly on A/K. Since $\dim H/K=1$ and H/K is a Cartan subalgebra of L/K, A/K is the Fitting one-component of L/K with respect to H/K. Hence $L_1+K/K=A/K$ and $L_1+K=A$. Furthermore, since $[K, L_1]\subseteq[H, L_1]\subseteq L_1$, $[K, L_1]\subseteq K\cap L_1=0$.

Let T be the subalgebra of L generated by L_1. Since L_1 is F-invariant, T is also, and since $L_1\subseteq A$, $T\subseteq A$. Now $F+T$ is a nonnilpotent subalgebra of L, hence $F+T=L$. Then, since T is F-invariant, $A=T$. If $K=0$, $L_1=A$ is a minimal ideal of L. Suppose then that $K\neq 0$. We claim that K is abelian. Since A is generated by L_1, we need only consider elements of K of the form $[x, y]$, $x, y\in L_1$. Using the Jacobi identity on elements of this type and that $[K, L_1]=0$, one sees that $[K, K]=0$. Then $[A, K]\subseteq[K, K]+[L_1, K]=0$ and $K\subseteq Z(A)$. If $K\subset Z(A)$, then, since $Z(A)$ is an ideal in L, $Z(A)$ is F-invariant and hence $Z(A)\cap L_1$ is F-invariant. But then $Z(A)=A$ which forces

$K=0$, a contradiction. Hence $Z(A)=K$. Since A/K is abelian, $A'\subseteq K$. But L_1 is a set of generators for A, therefore $K\subseteq A'$ and $A'=K$. Finally, since F acts irreducibly on A/K, the proof is complete.

The proof of Theorem 1 indicates we can state this result in the following way.

COROLLARY. *Let L be a nonnilpotent, solvable Lie algebra all of whose proper subalgebras are nilpotent. Let H be a self-normalizing maximal subalgebra of L, K be the core of H and $L=H+L_1$ be the Fitting decomposition of L with respect to H. Then L is the vector space direct sum of K, L_1 and F where F is a one-dimensional subalgebra of L and $K+F=H$. Furthermore $A=K+L_1$ is an ideal of L and either $K=0$ and L_1 is a minimal ideal of L or $A'=Z(A)=K$. In either case, F acts irreducibly and nontrivially on A/A'.*

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