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# MULTIPLICATIVE LINEAR FUNCTIONALS ON CONVOLUTION ALGEBRAS

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ABSTRACT. It is shown that semicharacters on the semigroup $S$ lead in a natural way to multiplicative linear functionals on $l(S)$, the convolution algebra of all complex valued functions on $S$. A theorem of D. H. Lehmer and a theorem of M. Tainiter follow as special cases.

1. **Introduction.** In [3] M. Tainiter proved and used a theorem with a long history. According to D. H. Lehmer [2] Daublebsky Von Sterneck stated and misproved a special case of the theorem in 1894. Lehmer generalized the theorem and correctly proved it in 1931. Lehmer's theorem can be further extended and we do so below in Theorem 4.

2. **Notation.**

2.1. $S$ will always denote a commutative semigroup with an identity, $e$, (we could omit the identity but keep it for convenience). Also, for each $n \in S$ the set $\{(x, y) \in S \times S \mid xy = n\}$ is finite.

2.2. Let $I = \{x \in S \mid x^2 = x\}$ be the set of idempotent elements in $S$.

2.3. Let $A_n = \{x \in S \mid xn = n\}$ be the set of associates of $n$.

2.4. Let $D_n = \{x \in S \mid \exists y \in S \text{ and } xy = n\}$ be the set of divisors of $n$. None of the sets $I$, $A_n$, and $D_n$ is empty.

2.5. Let $l(S)$ be the set of all complex-valued functions on $S$. Then $l(S)$ is a convolution algebra [4] with convolution given by

$$f * g(n) = \sum_{xy=n} f(x)g(y).$$

3. **Semicharacters.**

3.1. DEFINITION. Let $T$ be any semigroup. A semicharacter on $T$ is a bounded, multiplicative, complex-valued function $\chi$ on $T$ which is not identically zero. For our purposes we need another condition on $\chi$, i.e. $\{x \in T \mid \chi(x) \neq 0\}$ is finite. This is the only kind of semicharacter that we use, and we will refer to them simply as semicharacters.

4. THEOREM. *If $\chi$ is a semicharacter on $S$, then the map $f \to \sum_{x \in S} f(x)\chi(x)$ is a multiplicative linear functional on $l(S)$.*

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PROOF. Each sum below is finite so we rearrange the terms with impunity.

$$\left(\sum_{x\in S} f(x)\chi(x)\right)\left(\sum_{y\in S} g(y)\chi(y)\right) = \sum_{x,y\in S} f(x)g(y)\chi(xy)$$

$$= \sum_{n\in S}\left(\sum_{xy=n} f(x)g(y)\right)\chi(n) = \sum_{n\in S} f*g(n)\chi(n).$$

5. **Lehmer's theorem.** Let $N$ be the positive integers under any semigroup operation such that the conditions of §2 are satisfied with $e=1$. In addition Lehmer postulates

(5.1) $\qquad xyn \neq n \quad$ implies $xn \neq n$ and $yn \neq n$.

Equation (5.1) is enough to guarantee that $\chi_n$, the characteristic function of $A_n$ is a semicharacter on $N$.

For each $f\in l(N)$ define $F$ by

$$F(n) = \sum_{x\in A_n} f(x) = \sum_{x\in N} f(x)\chi_n(x).$$

Then if $f*g=h$, we have by Theorem 4 that $H(n)=F(n)G(n)$ which is Theorem 4 of [2].

6. **Tainiter's theorem.** Let $T$ be a finite, commutative semigroup of idempotents. If $x\in D_n$, then for some $y\in T$, $xy=n$ and $xyn=n^2=n$, i.e. $xy\in A_n$. It follows that $x\in A_n$ since $xn=x(xy)=x^2y=xy=n$. So (5.1) holds and Theorem 4 in this case is Theorem 3.1 of [3].

For applications of Theorem 4 to combinatorial analysis see Tainiter [3]. For application to a class of convolution algebras and to elementary number theory see [1].

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