HOMEOMORPHISM GROUPS OF WEAK SOLENOIDAL SPACES

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ABSTRACT. It is shown that inverse limit spaces of closed manifolds with covering maps as bonding maps are determined by their homeomorphism groups.

1. Introduction. A *weak solenoidal sequence* (*solenoidal sequence*) of closed manifolds is an inverse limit sequence (M, f) such that each factor space M_n is a closed manifold and each bonding map $f_m^n: M_n \rightarrow M_m$ is a covering map (regular covering map). The limit space M_∞ is called a *weak solenoidal space* (*solenoidal space*). If X is a topological space, let $G(X)$ denote the group of homeomorphisms of X onto itself. Give $G(X)$ the point-open topology and consider $G(X)$ as a topological space. The following theorem is our main result.

THEOREM. *Let M_∞ and N_∞ be weak solenoidal spaces, and let Φ be an isomorphism of $G(M_\infty)$ onto $G(N_\infty)$ which is also a homeomorphism. Then M_∞ and N_∞ are homeomorphic.*

A space X is *invertible* if, for each open set U , there is a homeomorphism h of X onto itself that maps $X - U$ into U . If h can be chosen to be isotopic to the identity, then we call X *continuously invertible*. Doyle and Hocking [2] have studied continuous near-homogeneity as a generalization of continuous invertibility. X is *continuously near-homogeneous* if given any point $p \in X$ and an open set U of X , there is a homeomorphism $h: X \rightarrow X$ such that $h(p) \in U$ and h is isotopic to the identity. We complete this paper by showing that a weak solenoidal space is invertible if and only if it is a solenoid.

If $P = (p_1, p_2, \dots)$ is a sequence of prime numbers (different from 1), the P -adic solenoid Σ_P is the limit of the inverse sequence (M, f) , where each $M_n = \{z: |z| = 1\}$ (the unit circle in the complex plane), and each bonding map $f_n^{n+1}: M_{n+1} \rightarrow M_n$ is defined by $f_n^{n+1}(z) = z^{p_n}$. The dyadic solenoid Σ_2 is the solenoid corresponding to the sequence $(2, 2, 2, \dots)$. The one-dimensional weak solenoidal spaces are precisely the solenoids. D. van Dantzig discussed solenoids in [1].

We follow the notation on inverse limit systems in [4] and restrict

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our discussion to inverse limit sequences (the directed set of indices is the positive integers). We assume that all weak solenoidal sequences are nontrivial, i.e., each bonding map is at least a 2-fold covering map. All spaces considered are compact, metric, and nondegenerate and all manifolds are connected. If $x \in X$, the *isotropy subgroup* at x is the group $G_x = \{h \in G(X) : h(x) = x\}$. If X is a topological group, then the map θ of the quotient space $G(X)/G_x$ into X , given by $\theta(gG_x) = g(x)$, is a continuous bijection. Let G and H be two groups which are also topological spaces (not necessarily topological groups). We say that G is *topologically isomorphic* to H if there exists an isomorphism of G onto H which is also a homeomorphism.

2. Invertibility and continuous near-homogeneity. The behavior of weak solenoidal spaces with respect to the properties of invertibility and continuous near-homogeneity closely parallels the behavior of closed manifolds. Doyle and Hocking [3] have observed that spheres are the only invertible closed manifolds, and it is easy to see that each closed manifold is continuously homogeneous. We show that solenoids (the only weak solenoidal spaces which are inverse limits of spheres) are the only invertible weak solenoidal spaces, and that each weak solenoidal space is continuously near-homogeneous.

THEOREM 1. *Let $M_\infty = \lim (M, f)$ be a weak solenoidal space. Then M_∞ is continuously near-homogeneous.*

PROOF. Let x and y be points of M_∞ , and let U be an open set in M_∞ containing x . There exists an open cell C in M_n (for some n) such that $x \in f_n^{-1}(C) \subset U$. Let W be a closed cell in M_n containing y_n and x_n in its interior. There is a homeomorphism $h_n : M_n \rightarrow M_n$ such that $h_n(y_n) = x_n$ and h_n is the identity on the complement of W . Since W is evenly covered by disjoint collections of closed cells, we may lift h_n up the inverse sequence to obtain the commutative diagram

$$\begin{array}{ccccccc} M_n & \leftarrow & M_{n+1} & \leftarrow & M_{n+2} & \leftarrow & \cdots \\ h_n \downarrow & & h_{n+1} \downarrow & & h_{n+2} \downarrow & & \\ M_n & \leftarrow & M_{n+1} & \leftarrow & M_{n+2} & \leftarrow & \cdots \end{array}$$

Let $h : M_\infty \rightarrow M_\infty$ be the homeomorphism induced by the sequence of homeomorphisms $\{h_n\}$. Then $(h(y))_n = x_n$, so $h(y) \in f_n^{-1}(x_n) \subset f_n^{-1}(C) \subset U$. Since h is isotopic to the identity map, M_∞ is continuously near-homogeneous. $\square$

Doyle and Hocking [3], considering the dyadic solenoid as the intersection of solid tori, have shown that it is invertible. This can be done more generally by an inverse limit argument.

THEOREM 2. *If M_∞ is a weak solenoidal space, then the following are equivalent:*

- (a) M_∞ is a solenoid;
- (b) M_∞ is continuously invertible;
- (c) M_∞ is invertible.

PROOF. Suppose M_∞ is a solenoid. To prove that M_∞ is continuously invertible, it suffices to construct a homeomorphism h of M_∞ onto itself that maps the complement of a basic open set U into U such that h is isotopic to the identity. U is of the form $f_n^{-1}[(a, b)]$, where (a, b) is an open segment in M_n .

Let h_n be a homeomorphism of M_n onto itself of degree 1 which maps the complement of (a, b) into (a, b) . Let J_n be an isotopy between h_n and the identity map. For each $m > n$, J_n lifts to an isotopy between the identity map of M_m and a homeomorphism $h_m: M_m \rightarrow M_m$. Hence J_n lifts to an isotopy J_∞ between the identity map of M_∞ and a homeomorphism $h: M_\infty \rightarrow M_\infty$ such that h maps the complement of U into U .

Suppose that M_∞ is invertible. Let U_1 be an open n -cell in M_1 , and let $U = f_1^{-1}(U_1)$ be a basic open set in M_∞ . U is homeomorphic to the product of U_1 and a Cantor set. For each n , $f_n(U)$ is a finite number of disjoint open n -cells in M_n . If M_∞ is not a solenoid, then $M_n - f_n(U)$ is compact and connected. Thus $M_\infty - U$ is connected and compact. Let h be a homeomorphism of M_∞ onto itself which maps $M_\infty - U$ into U . Then $h(M_\infty - U)$ is a subset of an n -cell. This contradiction completes the proof. $\square$

3. Homeomorphism groups. In this section we show that the homeomorphism groups of weak solenoidal spaces determine the spaces.

First we quote a theorem of E. S. Thomas, Jr. [8].

LEMMA 1. *Suppose that X_1 and X_2 are spaces, and suppose that $G(X_1)$ and $G(X_2)$ are topologically isomorphic. Then sufficient conditions that X_1 and X_2 be homeomorphic are (for $i = 1, 2$):*

Condition A. Let P and Q be finite disjoint sets, let y be a point of $X_i - P$, and let V be an open set in X_i . If some member of $G(X_i)$ maps y into V , then some member of $G(X_i)$ maps y into $V - Q$, leaving P point-wise fixed.

Condition B. If $\{x_n\}$ is a sequence not converging to $x \in X_i$, then there is a subsequence $\{y_n\}$ of $\{x_n\}$ and a map g in $G(X_i)$ such that $g(y_n) = y_n$ for all n and $g(x) \neq x$.

LEMMA 2. *If M_∞ is a weak solenoidal space which is not a solenoid, then M_∞ satisfies both Condition A and Condition B.*

PROOF. We notice that since M_∞ has dimension > 1 , W in the proof of Theorem 1 can be chosen so that $f_n^{-1}(W) \cap P = \emptyset$. Hence the constructed homeomorphism h leaves P pointwise fixed. Furthermore, we can choose the point x so that Q does not intersect $f_n^{-1}(x_n)$. Hence M_∞ satisfies Condition A.

Let $\{x_n\}$ be a sequence in M_∞ not converging to x . Let $\{y_n\}$ be a subsequence of $\{x_n\}$ which converges to a point y in M_∞ , $y \neq x$, such that $y_n \neq x$, for all n . Let U be an open set in M_∞ which contains x and does not contain any y_n . Let C be an open cell in M_k such that $x \in f_k^{-1}(C) \subset U$. Let h_k be a homeomorphism of M_k onto itself such that $h_k(f_k(x)) \neq f_k(x)$ and h_k is the identity outside U . For each $m > k$, h_k lifts to a homeomorphism $h_m: M_m \rightarrow M_m$. The sequence $\{h_m\}$ induces a homeomorphism h of M_∞ onto itself such that $h(x) \neq x$ and $h(y_n) = y_n$, for all n . Therefore, M_∞ satisfies Condition B. $\square$

REMARK. Solenoids satisfy Condition B but not Condition A.

REMARK. We have actually shown the stronger result that the group of induced homeomorphisms of a weak solenoidal space of dimension > 1 is a determining subgroup (see [8] for the definition of determining group and [7] for a proof that the induced homeomorphisms do form a group).

LEMMA 3. *Let S_1 and S_2 be solenoids. If $G(S_1)$ is topologically isomorphic to $G(S_2)$, then S_1 is homeomorphic to S_2 .*

PROOF. We can embed S_1 (as a topological group) in $G(S_1)$ as the group of translations. Hence S_1 can be embedded as a topological group in $G(S_2)$. Let $\gamma: S_1 \rightarrow G(S_2)$ denote this embedding.

Let x be a point in S_2 , and let $p: G(S_2) \rightarrow G(S_2)/G_x$ be the projection map. Only the identity of S_1 fixes x , so $\theta \circ p \circ \gamma$ is an injective map of the compact space S_1 into S_2 (recall $\theta: G(S_2)/G_x \rightarrow S_2$). Hence S_1 is topologically contained in S_2 . Therefore, S_1 is homeomorphic to S_2 , for a solenoid contains no proper indecomposable continua. $\square$

The next lemma is Theorem 3.1 of [8] with a weakening of the hypotheses and a corresponding weakening of the conclusion.

LEMMA 4. *Let F' and F be spaces, and let Φ be a topological isomorphism of $G(F')$ onto $G(F)$. If F satisfies Condition A, then, for each x in F' , $\Phi(G_x)$ is the isotropy subgroup G_y of some point y of F .*

LEMMA 5. *Let S be a solenoid, and let M_∞ be a weak solenoidal space.*

If $G(S)$ is topologically isomorphic to $G(M_\infty)$, then S is homeomorphic to M_∞ .

PROOF. If M_∞ is a solenoid, then the conclusion follows from Lemma 3. If M_∞ is not a solenoid, then M_∞ satisfies Condition A. Let Φ be a topological isomorphism of $G(S)$ onto $G(M_\infty)$. Let $x \in S$; then $\Phi(G_x) = G_y$, for some y in M_∞ .

Let U be a basic open set in M_∞ containing y . $\text{Cl}(U)$ is homeomorphic to the product of a Cantor set A with a closed cell B . There exists a simple triod T of homeomorphisms of $A \times B$ into itself such that $T \cap G_y = \{1\}$ and such that each homeomorphism agrees with the identity on $A \times \partial B$ (since such triods clearly exist for B). Extend T to all of M_∞ by letting each homeomorphism of T be fixed on the complement of $A \times B$.

Thus there exists a triod $\Phi^{-1}(T)$ in $G(S_1)$. Furthermore $\Phi^{-1}(T) \cap G_x$ is the identity map. Hence, if $p: G(S) \rightarrow G(S)/G_x$ is the projection map, then $\theta \circ p(\Phi^{-1}(T))$ is a triod. This is a contradiction, for each solenoid is atriodic. $\square$

Finally, combining Lemmas 1, 2, and 5, we obtain the promised result.

THEOREM 3. *Let M_∞ and N_∞ be weak solenoidal spaces. If $G(M_\infty)$ and $G(N_\infty)$ are topologically isomorphic, then M_∞ and N_∞ are homeomorphic.*

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