A NOTE ON STABLE HOMEOMORPHISMS OF INFINITE-DIMENSIONAL MANIFOLDS¹

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ABSTRACT. In papers by R. D. Anderson and R. Wong, respectively, it is shown that all homeomorphisms of the Hilbert cube onto itself, or of the infinite dimensional separable Hilbert space l_2 onto itself, are stable in the sense of Brown-Gluck. These facts can be used to show that all homeomorphisms of X onto itself are isotopic to the identity mapping where X is either the Hilbert cube or l_2 . It follows that some versions of the infinite-dimensional annulus conjecture are true. In this note we give a simple proof of Anderson's result. It follows from Brown-Gluck's technique that for any connected manifold X modeled on Q or s , every homeomorphism of X onto itself is stable.

An end-face K of the Hilbert cube $Q = [0, 1]^\infty$ is a set of the form $\pi_i^{-1}(0)$ or $\pi_i^{-1}(1)$ where π_i is the projection of Q onto the i th-factor. We define $s = (0, 1)^\infty \subset Q$.

Let $G(X)$ denote the set of all homeomorphisms of X onto X and let " $\approx$ " mean "homeomorphic to". A homeomorphism of X onto itself is *stable* if it is the finite composition of homeomorphisms each of which is the identity on some nonempty open subset of X .

LEMMA. *Let h be any homeomorphism of Q onto Q and let B be the union of all end-faces of Q , then there is an imbedding f of Q into s such that $f(Q) \cap h(B) = \emptyset$.*

PROOF. Write B as $K_1 \cup K_2 \cup \dots$ where each K_i is an end-face of Q . Let H denote the space of all imbeddings of Q into Q under the usual supremum metric. According to [5, p. 32] H is complete, hence is of 2nd-category. Let $H_i = \{f' \in H : f'(Q) \cap (K_i \cup h(K_i)) = \emptyset\}$. Clearly an $f \in \bigcap_{i=1}^\infty H_i$ is the desired function if such an f exists. Therefore it suffices to show that each H_i (namely H_1) is both open and dense in H . H_1 is clearly open. To show H_1 is dense, suppose $\epsilon > 0$ and $g \in H$. Consider $h^{-1}g(Q)$. It is clear there is a $g_1 \in H$ such that $g_1(h^{-1}g(Q)) \cap K_1 = \emptyset$. Hence $h[g_1h^{-1}g(Q)] \cap h(K_1) = \emptyset$. Again it is clear there is a $g_2 \in H$ such that $f' = g_2[hg_1h^{-1}g(Q)] \cap (K_1 \cup h(K_1)) = \emptyset$. We can now assume g_1, g_2 are to be chosen so small so that

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$(g_2)(hg_1h^{-1})g$ is arbitrarily closed to g or $d(f', g) < \epsilon$. This shows H_1 is dense and the Lemma is proved.

COROLLARY (ANDERSON [3]). *Every $h \in G(Q)$ is stable.*

PROOF. It follows from the Lemma that for some sub-Hilbert cube $K \subset s$, $h(K) \subset s$. By [1] there are $f, g \in G(Q)$ with f, g stable, f equal to h^{-1} on $h(K)$ and with $g(K) = K_1$, where $K_1 = \pi_1^{-1}(0)$. Then $gfhg^{-1}$ is the identity on K_1 . It follows that $gfhg^{-1}$ is stable (Wong proves it for s [6], similar result for Q , Anderson [3, Lemma 3.1]). Hence h is stable.

The author thanks the referee for pointing out an alternative (but simpler) proof of the following theorem, whose version is adopted here.

THEOREM. *Let X be a connected manifold modeled on Q or s , then every $h \in G(X)$ is stable.*

PROOF. Let $p \in X$. Following the standard device of forming simple chains from p to $h(p)$, there is a stable homeomorphism g of X onto itself such that $g(h(p)) = p$. Thus for simplicity we may assume $h(p) = p$. Let U, V be basic open sets in X with $p \in V \subset \text{Cl}(V) \subset U$, with $\text{Bd}(U) \approx \text{Bd } V \approx Q$ or s , and with $h(V) \subset U$. Then by Corollary 10.6 of [3] (for Q) and a similar argument for s (using [6]), $\text{Cl}(U) \setminus V \approx \text{Cl}(U) \setminus h(V) \approx Q$ (or s). But since $\text{Bd } U \cup \text{Bd } V, \text{Bd } U \cup \text{Bd } h(V)$ are Z -sets of $\text{Cl}(U) \setminus V$ and $\text{Cl}(U) \setminus h(V)$ respectively, by Theorem 10.1 of [3] there is a homeomorphism f of $\text{Cl}(U) \setminus h(V)$ onto $\text{Cl}(U) \setminus V$ which is the identity on $\text{Bd } U$ and is h^{-1} on $h(\text{Bd } V)$. Let F be the extension of f to X onto itself which is the identity on $X \setminus U$ and is h^{-1} on $h(V)$. Then Fh is the identity on V . Since $h = F^{-1}(Fh)$, h is stable.

REFERENCES

1. R. D. Anderson, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. **126** (1967), 200–216. MR **34** #5045.
2. ———, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519. MR **32** #8298.
3. ———, *On topological infinite deficiency*, Michigan J. Math. **14** (1967), 365–383. MR **35** #4893.
4. M. Brown and H. Gluck, *Stable structures on manifolds. I. Homeomorphisms of S^n* , Ann. of Math. (2) **79** (1964), 1–17. MR **28** #1608a.
5. C. Kuratowski, *Topologie*. Vol. 2, 3rd ed., Monografie Mat., Tom 21, PWN, Warsaw, 1961, p. 32 (7); English transl., Academic Press, New York (to appear). MR **24** #A2958.
6. R. Y. T. Wong, *On homeomorphisms of certain infinite dimensional spaces*, Trans. Amer. Math. Soc. **128** (1967), 148–154. MR **35** #4892.

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