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ON AN INTEGRAL FORMULA OF GAUSS-BONNET-GROTEMEYER

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ABSTRACT. Let $e(p)$ and $G(p)$ be the unit outer normal and the Gauss-Kronecker curvature of an oriented closed even-dimensional hypersurface M of dimension n in E^{n+1}. Then for a fixed unit vector c in E^{n+1}, we have

$$(1)\qquad \begin{aligned} \int_M (\mathbf{c}\cdot\mathbf{e})^m G\,dV &= c_{n+m}\chi(M)/c_m, &&\text{for } m = 0, 2, 4, \cdots, \\ &= 0, &&\text{for } m = 1, 3, 5, \cdots, \end{aligned}$$

where $\mathbf{c}\cdot\mathbf{e}$ denotes the inner product of c and e, c_m the area of m-dimensional unit sphere, and $\chi(M)$ the Euler characteristic of M.

Let M be an orientable closed hypersurface imbedded in a euclidean space E^{n+1} of dimension $n+1\geqq 3$. Let $\mathbf{x}(p)$ be the position vector of a point p with respect to a fixed point 0 in E^{n+1}, and $\mathbf{e}(p)$, $G(p)$ and dV the unit outer normal, the Gauss-Kronecker curvature at p, and the volume element of M in E^{n+1}, respectively. The main results of this paper are the following:

THEOREM 1. *Let M be an oriented closed hypersurface of dimension n imbedded in euclidean space E^{n+1} of dimension $n+1\geqq 3$. Then we have*

$$(2)\qquad m\int_M (\mathbf{x}\cdot\mathbf{e})^{m-1}\mathbf{x}G\,dV = (n+m)\int_M (\mathbf{x}\cdot\mathbf{e})^m \mathbf{e}G\,dV, \qquad m = 0, 1, 2, 3, \cdots.$$

THEOREM 2. *Under the same hypothesis of Theorem* 1, *if the dimension of M is even, then for a fixed unit vector* $\mathbf{c}$ *in E^{n+1}, we have*

$$(3)\qquad \begin{aligned} \int_M (\mathbf{c}\cdot\mathbf{e})^m G\,dV &= c_{n+m}\chi(M)/c_m, &&\textit{for } m = 0, 2, 4, \cdots, \\ &= 0, &&\textit{for } m = 1, 3, 5, \cdots. \end{aligned}$$

REMARK. If $m=0$, then formula (3) is the well-known Gauss-Bonnet formula, and if $m=2$ and $n=2$, then formula (3) was proved by Grotemeyer [3] in 1963.

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1. **Preliminaries.** Let M be an oriented (differentiable) manifold of dimension n, and let $x: M \to E^{n+1}$ be a hypersurface. Let $\mathbf{e}(p)$, $p \in M$, be the unit outer normal at $x(p)$. We consider the orthonormal frames $\mathbf{e}_1, \cdots, \mathbf{e}_n$ in the tangent hyperplane at $x(p)$, such that the determinant $(\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{e}) = +1$. The space of all $\mathbf{e}_1, \cdots, \mathbf{e}_n$ can be identified with the principal fibre bundle B of M relative to the induced metric $dx \cdot dx$ (for the details, see Chern [2]). We have

$$dx = \omega_1 \mathbf{e}_1 + \cdots + \omega_n \mathbf{e}_n, \qquad d\mathbf{e} = \theta_1 \mathbf{e}_1 + \cdots + \theta_n \mathbf{e}_n, \tag{4}$$

so that ω_i, θ_i, $1 \leqq i \leqq n$, are linear differential forms in B. Since

$$\mathbf{e} \cdot dx = 0, \tag{5}$$

we get, by exterior differentiation,

$$d\mathbf{e} \wedge dx = 0. \tag{6}$$

The left-hand side in (6) is the exterior product of two vector-valued linear differential forms; vectors are multiplied in the sense of scalar products in E^{n+1}. In view of (4), equation (6) can be written

$$\sum_i \omega_i \wedge \theta_i = 0. \tag{7}$$

Since ω_i are linear independent, we can put, in view of (7),

$$\theta_i = \sum A_{ij}\omega_j, \quad A_{ij} = A_{ji}, \qquad 1 \leqq i, j \leqq n. \tag{8}$$

The Gauss-Kronecker curvature G is given by

$$G = \det(A_{ij}). \tag{9}$$

Since $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is an orthonormal frame, we know that the volume element $dV = \omega_1 \wedge \cdots \wedge \omega_n$. Hence, by (8) and (9), we have

$$\theta_1 \wedge \cdots \wedge \theta_n = GdV. \tag{10}$$

For simplicity, let $[\ ,\ \cdots,\ ]$ (n terms) denote the combining operation of the vector product of E^{n+1} with the exterior product. From (10), we have

$$\overset{(n \text{ times})}{[d\mathbf{e}, \cdots, d\mathbf{e}]} = (n!GdV)\mathbf{e}. \tag{11}$$

2. **Proof of Theorem 1.** Put

$$\mathfrak{d} = \sum_i (-1)^{i-1}\theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_n \mathbf{e}_i, \tag{12}$$

where "^" denotes the omitted term. Then, from (4), we have

$$
\begin{aligned}
&\overset{(n-1 \text{ times})}{[de, \cdots, de, e]} = \overset{(n-1 \text{ times})}{[\sum \theta_i e_i, \cdots, \sum \theta_i e_i, e]} \\
(13) \quad &= (n-1)! \sum \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_n [e_1, \cdots, \hat{e}_i, \cdots, e_n, e] \\
&= (n-1)! \sum (-1)^{n-i-1} \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_n e_i \\
&= (n-1)! (-1)^n \sigma.
\end{aligned}
$$

From (11) and (12), we get

$$
d\sigma = -\overset{(n \text{ times})}{[de, \cdots, de]}/(n-1)! = -(nGdV)e. \tag{14}
$$

By (4), (12) and (14), we have

$$
\begin{aligned}
d((\mathrm{x}\cdot e)^m \sigma) &= m(\mathrm{x}\cdot e)^{m-1}(\mathrm{x}\cdot de) \wedge \sigma + (\mathrm{x}\cdot e)^m d\sigma \\
(15) \quad &= m(\mathrm{x}\cdot e)^{m-1} \sum (\mathrm{x}\cdot e_i) e_i \theta_1 \wedge \cdots \wedge \theta_n + (\mathrm{x}\cdot e)^m d\sigma \\
&= m(\mathrm{x}\cdot e)^{m-1} \mathrm{x} GdV - (n+m)(\mathrm{x}\cdot e)^m e GdV.
\end{aligned}
$$

Integrating both sides of (15) over M and applying Stokes' theorem, we get (2). This completes the proof of the theorem.

3. **Proof of Theorem 2.** Let c be a unit vector in E^{n+1}. Taking the scalar product of c with both sides of (2), we get

$$
m \int_M (\mathrm{x}\cdot e)^{m-1}(\mathrm{x}\cdot \mathrm{c}) GdV = (n+m) \int_M (\mathrm{x}\cdot e)^m (\mathrm{c}\cdot e) GdV. \tag{A_0}
$$

We make the translation $x \to x + \mathrm{c}$ of M. Then, by (A_0), we get

$$
\begin{aligned}
&m \int_M \sum_{i_1=0}^{m-1} \binom{m-1}{i_1} (\mathrm{x}\cdot e)^{i_1} (\mathrm{c}\cdot e)^{m-i_1-1}((\mathrm{x}\cdot \mathrm{c}) + 1) GdV \\
(A_0') \quad &\qquad = (n+m) \int_M \sum_{i_1=0}^{m} \binom{m}{i_1} (\mathrm{x}\cdot e)^{i_1} (\mathrm{c}\cdot e)^{m-i_1+1} GdV.
\end{aligned}
$$

$(A_0') - (A_0)$ gives

$$
\begin{aligned}
&m \int_M \sum_{i_1=0}^{m-2} \binom{m-1}{i_1} (\mathrm{x}\cdot e)^{i_1} (\mathrm{x}\cdot \mathrm{c}) (\mathrm{c}\cdot e)^{m-i_1-1} GdV \\
(A_1) \quad &\qquad + m \int_M \sum_{i_1=0}^{m-1} \binom{m-1}{i_1} (\mathrm{x}\cdot e)^{i_1} (\mathrm{c}\cdot e)^{m-i_1-1} GdV \\
&\qquad = (n+m) \int_M \sum_{i_1=0}^{m-1} \binom{m}{i_1} (\mathrm{x}\cdot e)^{i_1} (\mathrm{c}\cdot e)^{m-i_1+1} GdV.
\end{aligned}
$$

Again we make the translation $x \to x + \mathrm{c}$ of M into (A_1) and then subtract from (A_1), we get

$$m\int_M \sum_{i_1=0}^{m-2}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}(\mathbf{x}\cdot\mathbf{e})^{i_2}(\mathbf{x}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{e})^{m-i_2-1}GdV$$

$$\text{(A}_2\text{)}\qquad +m\int_M\left[\sum_{i_1=0}^{m-2}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}+\sum_{i_1=0}^{m-1}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}\right]\cdot(\mathbf{x}\cdot\mathbf{e})^{i_2}(\mathbf{c}\cdot\mathbf{e})^{m-i_2-1}GdV$$

$$=(n+m)\int_M \sum_{i_1=0}^{m-1}\binom{m}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}(\mathbf{x}\cdot\mathbf{e})^{i_2}(\mathbf{c}\cdot\mathbf{e})^{m-i_2+1}GdV.$$

Continuing this process k times ($k=1, 2, \cdots, m$), we get

$$m\int_M \sum_{i_1=0}^{m-2}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}\cdots\sum_{i_k=0}^{i_{k-1}-1}\binom{i_{k-1}}{i_k}\cdot(\mathbf{x}\cdot\mathbf{e})^{i_k}(\mathbf{x}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{e})^{m-i_k-1}GdV$$

$$+m\int_M\Bigg[\sum_{i_1=0}^{m-2}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}\cdots\sum_{i_{k-1}=0}^{i_{k-2}-1}\binom{i_{k-2}}{i_{k-1}}\sum_{i_k=0}^{i_{k-1}}\binom{i_{k-1}}{i_k}+\cdots$$

$$\text{(A}_k\text{)}\qquad +\sum_{i_1=0}^{m-2}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}\cdots\sum_{i_j=0}^{i_{j-1}}\binom{i_{j-1}}{i_j}\cdots\sum_{i_k=0}^{i_{k-1}-1}\binom{i_{k-1}}{i_k}+\cdots$$

$$+\sum_{i_1=0}^{m-1}\binom{m-1}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}\cdots\sum_{i_k=0}^{i_{k-1}-1}\binom{i_{k-1}}{i_k}\Bigg]\cdot(\mathbf{x}\cdot\mathbf{e})^{i_k}(\mathbf{c}\cdot\mathbf{e})^{m-i_k-1}GdV$$

$$=(n+m)\int_M \sum_{i_1=0}^{m-1}\binom{m}{i_1}\sum_{i_2=0}^{i_1-1}\binom{i_1}{i_2}\cdots\sum_{i_k=0}^{i_{k-1}-1}\binom{i_{k-1}}{i_k}\cdot(\mathbf{x}\cdot\mathbf{e})^{i_k}(\mathbf{c}\cdot\mathbf{e})^{m-i_k+1}GdV.$$

In particular, if $k=m$, then the first integral of (A$_k$) does not appear, and the terms in [$*+\cdots+*$] in the second integrand is equal to m!. Thus, (A$_m$) gives us the following formula:

$$\text{(16)}\qquad m\int_M \text{[illegible]}(\mathbf{c}\cdot\mathbf{e})^{m-1}GdV=(n+m)\int_M(\mathbf{c}\cdot\mathbf{e})^{m+1}GdV,\qquad m=1, 2, 3, \cdots.$$

Hence, we get

$$\int_M (\mathbf{c}\cdot\mathbf{e})^m G\,dV = \frac{m-1}{n+m-1}\int_M (\mathbf{c}\cdot\mathbf{e})^{m-2} G\,dV. \tag{17}$$

By the assumption, n is even. Hence if m is a positive even integer, then by (17), the Gauss-Bonnet formula and the fact

$$c_N = 2[\Gamma(\tfrac{1}{2})]^{N+1}/\Gamma(\tfrac{1}{2}(N+1)), \tag{18}$$

we get

$$\begin{aligned}\int_M (\mathbf{c}\cdot\mathbf{e})^m G\,dV &= \frac{(m-1)(m-3)\cdots 1}{(n+m-1)(n+m-3)\cdots(n+1)}\int_M G\,dV \\ &= c_{n+m}\chi(M)/c_m.\end{aligned} \tag{19}$$

Moreover, by (2), we get

$$\int_M \mathbf{e}G\,dV = 0. \tag{20}$$

Taking the inner product of $\mathbf{c}$ with (20), we get

$$\int_M (\mathbf{c}\cdot\mathbf{e}) G\,dV = 0. \tag{21}$$

Hence, in view of (16) and (21), we find that

$$\int_M (\mathbf{c}\cdot\mathbf{e})^m G\,dV = 0, \qquad \text{for all } m = 1, 3, 5, \cdots. \tag{22}$$

Therefore, by (19), (22) and the Gauss-Bonnet formula, we get formula (3). This completes the proof of the theorem.

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