

CONTINUOUS AND PROPER DECOMPOSITIONS

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ABSTRACT. If X is a locally connected, locally peripherally compact Hausdorff space and if R is an equivalence relation on X with fibers which are connected with compact boundaries, then it is shown that three definitions of continuity of R are equivalent. Some of the propositions used to obtain this result are then applied to get sufficient conditions for a decomposition of certain types of metric spaces to be proper.

1. Introduction. The notion of continuity for decompositions of topological spaces goes back to R. L. Moore [6] and Alexandroff [1]. In recent years this concept and that of a proper decomposition have been quite useful in analysis as will be noted, for example, in [2], [4], [5], [7]. There are several definitions of continuity which are, in general, not equivalent and one purpose of this note is to give conditions under which they are equivalent. We also make some observations concerning proper decompositions. Essential use is made of the results and methods in [3] and [9].

2. Definitions and terminology. Consider an equivalence relation R on a topological space X . Given a set $S \subset X$, $R(S)$ is the set of all points $x \in X$ such that x is equivalent to some point of S and is called the saturation of S . In particular, if $x \in X$, then $R(x)$ is called a fiber. If F is a filterbase on X then R generates a filterbase, F_R , each element of which is the saturation of some element of F . The set of accumulation points of a filterbase F will be denoted by $A(F)$.

A relation R is *open* (*closed*) if the saturation of every open (closed) set in X is open (closed). R is *Hausdorff* if the quotient space is Hausdorff. R is *proper* if the saturation of every compact set is compact.

3. Continuity conditions. Let X, R be given and let $p: X \rightarrow X/R$ be the natural projection. Then consider the conditions:

- I. R is open.
- II. R is closed.

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III. For every filterbase F converging to $x \in X$, $R(x) \subset A(F_R)$.

IV. For every filterbase F converging to $x \in X$, $A(F_R) \subset R(x)$.

If R satisfies I (II) it is called lower (upper) semicontinuous in the open set sense. Condition II is obviously equivalent to:

II'. For every open set $U \subset X$ the union of all fibers contained in U is open.

If R satisfies III (IV) it is called lower (upper) semicontinuous in the limit sense.

Consider now the conditions:

C_1 : R satisfies I and II.

C_2 : R satisfies III and IV.

C_3 : (1) R is open, (2) R is semiclosed, i.e., the saturation of every compact set is closed and (3) if $U \subset X/R$ is an open connected set then each component of $p^{-1}(U)$ maps onto U .

C_1 is continuity in the open set sense, C_2 is continuity in the limit sense and C_3 was proposed by Whyburn and studied in [9] and [10].

It is well known (see [3] or [4]) that I and III are equivalent conditions and that C_2 holds if and only if R is open and Hausdorff. For first countable spaces C_2 is equivalent to: for every sequence $\{x_n\}$, $\{x_n\} \rightarrow x$ implies $R(x_n) \rightarrow R(x)$. Also, for first countable spaces condition IV is equivalent to:

IV'. Any fiber which intersects the limit inferior of a sequence of fibers contains the limit superior of this sequence.

Using the above remarks and [9, Theorem 1.1] we have the following result.

PROPOSITION 1. *If X is first countable and R is open and semiclosed, then R is C_2 .*

As a corollary to Proposition 1 we get the well known fact that if X is a first countable space in which compact subsets are closed, then X is Hausdorff.

PROPOSITION 2. *If X is a locally connected Hausdorff space and R is C_1 , then R is C_3 .*

PROOF. Since X is Hausdorff and R is closed, the saturation of every compact set is closed.

Suppose U is some region in X/R and Q is a component of $p^{-1}(U)$ which does not map onto U . Then there is a point $y \in U$ which is a limit point of $p(Q)$ with $y \in U - p(Q)$. Let $V \subset X$ be the union of the remaining components of $p^{-1}(U)$. V is then open and contains $p^{-1}(y)$ and hence V contains an open saturated set V' which contains $p^{-1}(y)$. Thus $p(V') \subset U$ is an open set which contains y but $p(V') \cap p(Q) = \emptyset$.

The proof of the following proposition is similar to that of [9, Theorem 7.1] and will be omitted.

PROPOSITION 3. *Let X be a locally connected, locally peripherally compact Hausdorff space and suppose R is C_3 with fibers which have compact boundaries. Then R is closed and hence C_1 .*

We can now give conditions which imply the equivalence of the three continuity definitions.

THEOREM. *If X is a locally connected, locally peripherally compact Hausdorff space and the fibers of R are connected with compact boundaries, then C_1 , C_2 , and C_3 are equivalent.*

PROOF. The equivalence of C_1 and C_2 is proved in [3, Satz 7] and the equivalence of C_1 and C_3 follows from Propositions 2 and 3.

The connectedness is necessary as is shown by the following well-known example.

EXAMPLE. Let $S \subset E^2$ be the set

$$S = \{(x, y) \mid y = 0, 0 \leq x \leq 1\} \cup \{(x, y) \mid y = x, 0 \leq x < 1/2\}.$$

Two points are R equivalent if they lie on the same vertical line. Then R is C_2 with compact fibers but not C_1 or C_3 .

The point-inverse decomposition of the complex exponential function is an example which is C_2 and C_3 but not C_1 .

4. Proper decompositions. If R is a closed equivalence relation with compact fibers, then R is proper. We also note that a fiber of an open and semiclosed relation is either an open set or is its own boundary.

PROPOSITION 4. *If X is connected, locally peripherally compact and metric and if R (nondegenerate) is open and semiclosed with fibers which are connected with compact boundaries, then R is proper.*

PROOF. Since R is open and semiclosed and X is connected it is easy to see that the fibers of R are compact. That R is closed follows from Proposition 1 and [3, Satz 7].

PROPOSITION 5. *If X is connected, locally connected, locally peripherally compact and metric and if R (nondegenerate) is C_3 with fibers which have compact boundaries, then R is proper.*

PROOF. The proof is similar to that of Proposition 4 except that here we use Proposition 3.

In both cases X/R is metric by the results of [8].

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