PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ON ROOTS AND SUBSEMIGROUPS OF NILPOTENT GROUPS

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ABSTRACT. E_ω, U_ω and D_ω semigroups are defined by extrapolating the definitions of their group counterparts; and a class n semigroup is defined to be a subsemigroup of a class n group. The purpose of this paper is to show that a class n E_ω semigroup generates an E_ω group and that a class n semigroup is U_ω if and only if it generates a U_ω group.

E_ω, U_ω, and D_ω semigroups are defined in a manner similar to the group definitions presented by G. Baumslag [1]. Thus, let ω be a nonempty fixed set of primes. A semigroup S is an E_ω semigroup if for any $s \in S$ and $p \in \omega$ there exists $t \in S$ so that $s = t^p$. S is a U_ω semigroup if pth roots are unique when they exist. If S is both E_ω and U_ω, then S is called a D_ω semigroup.

The properties of existence and uniqueness of roots are investigated with respect to subsemigroups of class n nilpotent groups (referred to as class n semigroups) and the subgroups which these semigroups generate.

The purpose of this paper is to prove the following two theorems:

THEOREM A. *A class n semigroup is a U_ω semigroup if and only if it generates a U_ω group.*

THEOREM B. *A class n E_ω semigroup generates an E_ω group.*

REMARK 1. The converse of Theorem B is not available. Consider, for example, the semigroup Q' of rational numbers q, $q \geqq 1$, under addition. Q' is not divisible, but it generates the divisible group of additive rationals.

REMARK 2. Referring to the paper of B. H. Neumann and T. Taylor [3] on subsemigroups of nilpotent groups, Theorems A and B may be recast as follows:

THEOREM A′. *A cancellative semigroup satisfying the L_n law is U_ω if and only if it generates a class n U_ω group.*

THEOREM B′. *An E_ω cancellative semigroup satisfying the L_n law generates a class n E_ω group.*

Received by the editors April 15, 1970.

AMS 1969 *subject classifications*. Primary 2040, 2092, 2093.

Key words and phrases. E_ω semigroup, U_ω semigroup, D_ω semigroup, ω-free group, ω-subgroup.

PROOF OF THEOREM A.

A1. *If S is a class n U_ω semigroup in a group G, then the center of the group generated by S, $Z(\mathrm{gp}\{S\})$, is ω-free i.e. having no elements of order p for any $p \in \omega$.*

According to Neumann-Taylor [3], $\mathrm{gp}\{S\} = SS^{-1} = S^{-1}S$. Thus, suppose $s_1 s_2^{-1} \in Z(\mathrm{gp}\{S\})$ for some $s_1, s_2 \in S$ and that $(s_1 s_2^{-1})^p = 1$, where p is some element of ω and 1 is the identity of G. Now, s_1 and s_2 commute so that $s_1^p = s_2^p$. Since S is U_ω, $s_1 = s_2$ and $s_1 s_2^{-1} = 1$.

A2. *If S is a class n U_ω semigroup in a group G, then $\mathrm{gp}\{S\}$ is ω-free.*

Suppose the upper central series for $\mathrm{gp}\{S\}$ is $\{1\} = Z_0 \leqq Z_1 \leqq \cdots \leqq Z_n = \mathrm{gp}\{S\}$. Z_1 is the center of $\mathrm{gp}\{S\}$ and, according to part A1, is ω-free. Since Z_1 is abelian, it is clearly a U_ω subgroup so that the identity subgroup Z_0 is an ω-subgroup of Z_1. Here we may recall that a subgroup H of a group K is called an ω-subgroup of K if the relation $k^p \in H$ implies $k \in H$ for any pair k and p, with $k \in K$ and $p \in \omega$.

Let us proceed by induction and assume for each pair $\{Z_{i-1}, Z_i\}$, $i = 1, \cdots, k-1$, that Z_i is ω-free and Z_{i-1} is an ω-subgroup of Z_i. It may then be shown that for $i = k$, the pair $\{Z_{k-1}, Z_k\}$ fulfills the conditions just described. Following the usual argument we conclude that $Z_n = \mathrm{gp}\{S\}$ is ω-free.

Now, suppose $s \in Z_k$ and $s^p = 1$ for some $p \in \omega$. Then $1 = [s^p, t]$ for each $t \in \mathrm{gp}\{S\}$. $[s^p, t] = [s^{p-1}, t]^s [s, t] = [s^{p-1}, t][[s^{p-1}, t], s][s, t]$. However, $[s^{p-1}, t] \in Z_{k-1}$ so that $[[s^{p-1}, t], s] \in Z_{k-2}$. Thus, $1 = [s^p, t] = [s^{p-1}, t][s, t]s_1$ where $s_1 \in Z_{k-2}$. Proceeding in this way we finally have $1 = [s^p, t] = [s, t]^p s_2$ for some $s_2 \in Z_{k-2}$. But, under the induction assumption, Z_{k-2} is an ω-subgroup of Z_{k-1} so that $[s, t] \in Z_{k-2}$ and s must now be an element of Z_{k-1}, which is assumed to be ω-free. Hence, $s^p = 1$ implies $s = 1$ and Z_k is ω-free.

We follow a similar argument to show that Z_{k-1} is an ω-subgroup of Z_k.

A3. *A class n semigroup is U_ω if and only if $\mathrm{gp}\{S\}$ is ω-free.*

Part A2 tells us that $\mathrm{gp}\{S\}$ is ω-free if S is a U_ω semigroup. On the other hand, suppose S is not U_ω. Then there exist $x, y \in S$, $x \neq y$, and $p \in \omega$ with $x^p = y^p$. But, following a statement of P. Hall [2], xy^{-1} has order dividing a power of p. Consequently, $\mathrm{gp}\{S\}$ is not ω-free.

The following corollary of A3 is an important statement on U_ω groups and is attributed to Mal′cev and Cernikov in Baumslag [1].

A4. *A nilpotent group G is a U_ω group if and only if it is ω-free.*

A5. THEOREM A. *A class n semigroup is a U_ω semigroup if and only if it generates a U_ω group.*

It is clear that if S is a subsemigroup of a U_ω group then S is also U_ω. The converse is obtained as a consequence of parts A3 and A4.

PROOF OF THEOREM B.

B1. *If S is an E_ω subsemigroup of a commutative group, then $H=\mathrm{gp}\{S\}$ is E_ω.*

Since $\mathrm{gp}\{S\}=SS^{-1}$, consider $xy^{-1}\in\mathrm{gp}\{S\}$ for some x, $y\in S$ and fix $p\in\omega$. Then, there exist x_1 and y_1 belonging to S so that $x_1^p=x$ and $y_1^p=y$. Thus, $xy^{-1}=(x_1y_1^{-1})^p$ and the conclusion follows.

B2. THEOREM B. *A class n E_ω semigroup generates an E_ω group.*

Having verified the theorem in the abelian case in part B1, we proceed by induction and assume for class $n>1$ that the conclusion is valid for E_ω semigroups of class less than n.

Let $H=\mathrm{gp}\{S\}$ have lower central series $H=H^1\geqq H^2\geqq\cdots\geqq H^{n+1}=\{1\}$. H/H^n and, consequently, S/H^n have class less than n. S/H^n is clearly E_ω so that by the induction assumption H/H^n is an E_ω group. Thus, for $h\in H$ and $p\in\omega$, there exists $h_1\in H$ so that $hH^n=(h_1H^n)^p$. It follows that $h=h_1^pz$ for some $z\in H^n$. We know that H^n is generated by all transforms in H of commutators of the form $[s_1,\cdots,s_n]$, where $s_i\in S$, $i=1,\cdots,n$, [2].

Consider the commutator mentioned above. There is a $t\in S$ so that $s_n=t^p$. Thus, $[s_1,\cdots,s_{n-1},s_n]=[s_1,\cdots,s_{n-1},t^p]=[s_1,\cdots,s_{n-1},t]^p$. We see that every transform of the commutators under consideration has pth roots in H^n. But H^n is also central and thus every element of H^n has a pth root in H^n.

Let $z_1\in H^n$ be a pth root of z. Then $h=h_1^pz=(h_1z_1)^p$; and H is an E_ω group.

Theorems A and B yield the following:

COROLLARY C. *If S is a class n D_ω semigroup then S generates a D_ω group.*

COROLLARY C′. *If S is a cancellative D_ω semigroup satisfying the L_n law, then S generates a class n D_ω group.*

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