

A REMARK ON MAHLER'S COMPACTNESS THEOREM

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ABSTRACT. We prove that if G is a semisimple Lie group without compact factors, then for all open sets $U \subset G$ containing the unipotent elements of G and for all $C > 0$, the set of discrete subgroups $\Gamma \subset G$ such that

(a) $\Gamma \cap U = \{e\}$,

(b) G/Γ compact and measure $(G/\Gamma) \leq C$,

is compact. As an application, for any genus g and $\epsilon > 0$, the set of compact Riemann surfaces of genus g all of whose closed geodesics in the Poincaré metric have length $\geq \epsilon$, is itself compact.

Consider the following general problem: let G be a locally compact topological group and let

$$\mathfrak{M}_G = \{\text{the set of discrete subgroups } \Gamma \subset G\}.$$

We would like to put a good topology on \mathfrak{M}_G and we would like to find fairly "big" subsets of \mathfrak{M}_G that turn out to be compact. Mahler studied the case $G = \mathbf{R}^n$, G/Γ compact, i.e., Γ is lattice (cf. Cassels [1, Chapter 5]). In this case, the group of automorphisms of G , $GL(n, \mathbf{R})$, acts transitively on the set of lattices, so that the subset $\mathfrak{M}_G^c \subset \mathfrak{M}_G$ of lattices can be identified as a homogeneous space under $GL(n, \mathbf{R})$; in fact:

$$\mathfrak{M}_G^c \cong GL(n, \mathbf{R})/GL(n, \mathbf{Z}).$$

So there is only one natural topology on \mathfrak{M}_G^c and Mahler's theorem states that for all ϵ and K :

$$\left\{ \Gamma \subset \mathbf{R}^n \left| \begin{array}{l} (1) \text{ if } \gamma \in \Gamma, \|\gamma\| < \epsilon \Rightarrow \gamma = 0 \\ (2) \text{ volume } (\mathbf{R}^n/\Gamma) \leq K \end{array} \right. \right\} \text{ is compact.}$$

(Cassels [1, p. 137].)

Chabauty [2] has investigated generalizations of Mahler's theorem to general G and subgroups Γ such that measure $(G/\Gamma) < +\infty$.¹ We topologize \mathfrak{M}_G by taking as a basis for the open sets the following:

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¹ Although in recent years this restriction has been commonly made by people investigating automorphic functions in several variables, in the classical cases it eliminates the Fuchsian groups $\Gamma \subset SL(2; \mathbf{R})$ of 2nd kind, and it eliminates all Kleinian groups $\Gamma \subset SL(2; \mathbf{C})$. And \mathfrak{M}_G seems very interesting in these cases.

- (1) $U \subset G$ open, $S_U = \{\Gamma \in \mathfrak{M}_G \mid \Gamma \cap U \neq \emptyset\}$,
 (2) $K \subset G$ compact, $T_K = \{\Gamma \in \mathfrak{M}_G \mid \Gamma \cap K = \emptyset\}$.

Then assuming that G is not too pathological,² Chabauty proves:

THEOREM. *Let U be an open neighborhood of e , C a positive number. Then: $\{\Gamma \in \mathfrak{M}_G \mid \Gamma \cap U = \{e\} \text{ and } \text{measure}(G/\Gamma) \leq C\}$ is compact.*

This is very pretty. Its main drawback, however, is that the topology on \mathfrak{M}_G is so weak that it is hard to deduce things from convergence in this topology. For instance if subgroups Γ_i converge to Γ , one would like to know that suitable sets of generators of the Γ_i converge to generators of Γ . Chabauty gives some arguments about this at the end of his paper, but I believe his reasoning there is wrong. However the results of Weil [4] and Macbeath [5] show that the topology is "strong enough" on the subset

$$\mathfrak{M}_G^c = \{\Gamma \in \mathfrak{M}_G \mid G/\Gamma \text{ compact}\}.$$

THEOREM (MACBEATH [5, THEOREMS 4 AND 5]). *Assume that G is a Lie group.³ Let subgroups $\Gamma_i \in \mathfrak{M}_G^c$ converge to $\Gamma \in \mathfrak{M}_G^c$. Then for i sufficiently large, there exist isomorphisms of the abstract groups*

$$\phi_i: \Gamma \xrightarrow{\sim} \Gamma_i$$

such that for all $\gamma \in \Gamma$, $\phi_i(\gamma) \in G$ converge to γ . Moreover there is a compact set $K \subset G$ and an open neighborhood $U \subset G$ of e such that $K \cdot \Gamma = G$, $K \cdot \Gamma_i = G$, $U \cap \Gamma = \{e\}$ and $U \cap \Gamma_i = \{e\}$ if i is sufficiently large.

For the application that we want, Chabauty's theorem is not the right generalization of Mahler's theorem. Instead, what we want is this:

THEOREM 1. *Let $G \subset GL(n, \mathbb{R})$ be a semisimple Lie group without compact factors. Let $U \subset G$ be an open set containing all unipotent elements of G and let C be a positive number. Then*

$$\{\Gamma \in \mathfrak{M}_G^c \mid \Gamma \cap U = \{e\}, \text{measure}(G/\Gamma) \leq C\}$$

is compact.

PROOF. This is an immediate consequence of Chabauty's theorem and Selberg's conjecture, proved recently by Kajdan and Margulis

² G satisfies the 2nd axiom of countability, and moreover $e \in G$ has a fundamental system of neighborhoods U_i such that $\text{measure}(\overline{U_i} - U_i) = 0$. In this case, \mathfrak{M}_G satisfies the 2nd axiom of countability too.

³ A Lie group is always assumed to be connected.

[3], to the effect that a discrete subgroup $\Gamma \subset G$, G as above, such that $\text{measure}(G/\Gamma) < +\infty$ but G/Γ not compact, must contain non-trivial unipotent elements of G . Q.E.D.

Instead of invoking the difficult result of Kařdan and Margulis, we can prove a weaker but more explicit theorem by elementary means: Let $G \subset GL(n, \mathbf{R})$ again be a semisimple Lie group without compact factors. Let $K \subset G$ be a maximal compact subgroup and let $X = K \backslash G$ be the associated symmetric space. Let the Killing form on G induce a metric ρ on X as usual. Define a function d on G by:

$$d(x) = \inf_{z \in X} \rho(z, z^x).$$

It is easy to see that d is continuous and $d(x) = 0$ if and only if when you decompose $x = x_s \cdot x_u$, (x_s semisimple, x_u unipotent and $x_s x_u = x_u x_s$), then x_s is in a compact subgroup of G or equivalently $x_s \in \bigcup_{y \in G} yKy^{-1}$. For all $\epsilon > 0$, define an open subset of G by:

$$U_\epsilon = \{x \in G \mid d(x) < \epsilon\}.$$

For all $C > 0$, define a compact subset of G by:

$$K_C = \{x \in G \mid \rho(K \cdot x, K \cdot e) \leq C\}.$$

THEOREM 2. *Let $n = \dim K \backslash G$. Then there is a constant γ depending only on n such that for all $\Gamma \in \mathfrak{M}_G^C$, $\epsilon > 0$,*

$$\Gamma \cap U_\epsilon = \{e\} \Rightarrow K_C \cdot \Gamma = G$$

where $C = \gamma \cdot \text{measure}(G/\Gamma) / \epsilon^{n-1}$. Hence for all positive D

$$\{\Gamma \in \mathfrak{M}_G \mid \Gamma^C \cap U_\epsilon = \{e\}, \text{measure}(G/\Gamma) \leq D\}$$

is compact.

PROOF. We begin by proving:

LEMMA. *Let X be a compact Riemannian manifold with all sectional curvatures $R(S) \leq 0$. There is a constant γ depending only on $n = \dim X$ such that:*

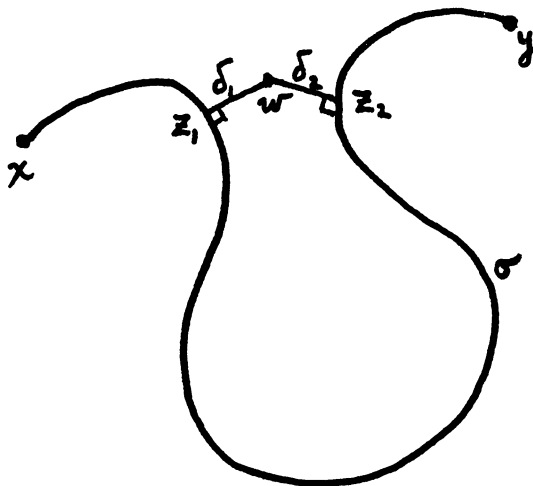
$$\text{diam}(X) \cdot (\text{length of smallest closed geodesic on } X)^{n-1} \leq \gamma \cdot \text{volume}(X).$$

PROOF. Let $d = \text{diam}(X)$ and let $x, y \in X$ be a distance d apart. Let σ be a geodesic from x to y of length d . Let η be the length of the shortest closed geodesic on X and construct a tube T around σ of radius $\eta/4$ as the union of all geodesics perpendicular to σ of length $\eta/4$. There are 2 possibilities: either no 2 geodesics δ_1, δ_2 perpendicular to σ of length $\eta/4$ meet, or else some pair δ_1, δ_2 do meet. In the first

case, we may say that the exponential map from the normal bundle N to σ in M maps an $\eta/4$ -tube T_0 around the 0-section in N injectively to M . Then since all the sectional curvatures are ≤ 0 , it follows that:

$$(*) \quad \text{volume } X \geq \text{volume } T \geq \text{volume } T_0 = c_n \cdot (\eta/4)^{n-1} \cdot d$$

where c_n is the volume of the unit ball in \mathbb{R}^{n-1} . On the other hand, suppose 2 geodesics δ_1 and δ_2 meet:



Let z_1 , z_2 and w be the points indicated in the figure and let e be the distance from z_1 to z_2 along σ . Then we can go from x to y by going from x to z_1 on σ , following δ_1 , then δ_2 and going from z_2 to y on σ . This has length $\leq d - e + \eta/2$, and since σ is the shortest path from x to y , $d \leq d - e + \eta/2$, i.e., $e \leq \eta/2$. But then δ_1 , δ_2 and the part of σ between z_1 and z_2 is a closed path τ of length at most η . τ is certainly not homotopic to 0 since on the universal covering space \tilde{X} of X , the exponential from N_0 to \tilde{X} is injective. Moreover, τ has corners and so is not itself a geodesic. Therefore there is a closed geodesic freely homotopic to τ of length $< \eta$. This contradicts the definition of η and so the 1st possibility must be correct. This proves $(*)$ and the lemma. Q.E.D.

We apply the lemma to the manifold X/Γ , with the metric induced from the metric d on X . (Note that by hypothesis $\Gamma \cap U_e = \{e\}$, Γ acts freely on X , so X/Γ is a manifold.) The closed geodesics of X/Γ are all images of geodesics in X joining 2 points x, x^z , where $x \in X$,

$z \in \Gamma$. Since $\Gamma \cap U_\epsilon = \{e\}$, these all have length at least ϵ . It follows from the lemma that:

$$\text{diam}(X) \leq \frac{\gamma \text{ volume } (X/\Gamma)}{\epsilon^{n-1}} = \frac{\gamma \text{ measure } (G/\Gamma)}{\epsilon^{n-1}} = C.$$

Therefore the projection of X onto X/Γ maps the unit ball of radius C onto X/Γ , hence $K_C \cdot \Gamma = G$.

Finally to prove from this that $\{\Gamma \in \mathfrak{M}_g^c \mid \Gamma \cap U_\epsilon = \{e\} \text{ and } \text{measure } (G/\Gamma) \leq D\}$ is compact, it suffices by Chabauty's theorem to check that if Γ_i are in this set and $\Gamma_i \rightarrow \Gamma \in \mathfrak{M}_g$, then G/Γ is also compact. But since $K_C \cdot \Gamma_i = G$ for all i , it follows easily that $K_C \cdot \Gamma = G$ too, hence G/Γ is a quotient of K_C and is compact. Q.E.D.

I want to apply Theorem 2 to the case $G = \text{SL}(2, \mathbf{R})/(\pm I)$ so that Γ is a Fuchsian group. Then X is the Lobachevskian plane, and a simple calculation shows that

$$\begin{aligned} U_\epsilon &= \text{image of } A \text{'s such that } |\text{tr } A| < 2 \cosh(\epsilon/2) \\ &= \text{set of elliptic and parabolic elements and those hyperbolic} \\ &\quad \text{elements with eigenvalues } t, t^{-1} \text{ for which } 1 < t < e^{\epsilon/2}. \end{aligned}$$

The Fuchsian groups of 1st kind which are disjoint from some U_ϵ are exactly those which act freely on X and for which X/Γ is compact. In this case X/Γ is a compact Riemann surface with its Poincaré metric, X is its universal covering space and $\Gamma \cong \pi_1(X/\Gamma)$. Moreover the map which takes an element $z \in \Gamma$ to the image mod Γ of the shortest line segment geodesic from x to x^z in X defines an isomorphism between the set of conjugacy classes in Γ and the set of closed geodesics in X/Γ . If the conjugacy class of γ corresponds to a geodesic σ , then

$$\cosh \frac{\text{length } \sigma}{2} = \left| \frac{\text{Tr } \gamma}{2} \right|.$$

Moreover, by the Gauss-Bonnet theorem

$$\text{measure } (G/\Gamma) = \text{area } (X/\Gamma) = \text{cnst } (g - 1)$$

where g = genus of X/Γ . So in this case, the lemma in Theorem 2 says:

COROLLARY 1. *For all compact Riemann surfaces X of genus g , $\text{diam}(X) \cdot (\text{length of smallest geodesic on } X)$ is bounded above.*

COROLLARY 2. *For all $\epsilon > 0$, $g \geq 2$, the set of discrete subgroups $\Gamma \subset \text{SL}(2; \mathbf{R})$ such that:*

- (i) for all $\gamma \in \Gamma$, $\gamma \neq I$, $|\operatorname{Tr} \gamma| \geq 2 + \epsilon$,
 (ii) X/Γ is a compact Riemann surface of genus g ,
 is compact.

COROLLARY 3. Let $g \geq 2$ and let \mathfrak{M}_g be the moduli space of compact Riemann surfaces of genus g (without "marking"). For all $\epsilon > 0$, the subset:

$$\{X \in \mathfrak{M}_g \mid \text{in the Poincaré metric, all geodesics on } X \text{ have length} \geq \epsilon\}$$

is compact.

(PROOF. Apply Theorem 1 and Corollary 1.)

This result was my motivation for looking at these questions. I originally found a completely elementary proof of this, using the method of Theorem 2, and then finding

- (a) upper bounds for the number of vertices and
 (b) lower bounds for the interior and exterior angles of the *Dirichlet* fundamental domain for Γ acting on X ; but one reference leads to another and it turned out that $\{\text{elem. th.}\} \subset \text{Chabauty} + \text{Weil} + \text{Každan-Margulis} + \text{Macbeath}$.

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