PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ANALYTIC PROPERTIES OF ELLIPTIC AND CONDITIONALLY ELLIPTIC OPERATORS[1]

MICHAEL E. TAYLOR

ABSTRACT. In this note we give a short proof of a theorem of Kotake and Narasimhan to the effect that if A is a strongly elliptic operator of order $2m$ with analytic coefficients and $\|A^j u\| \leqq C^{j+1}(2mj)!$, where $\|\ \|$ is some suitable norm, then u is analytic. (Actually Kotake and Narasimhan prove the theorem when A is elliptic, but the trick we use here requires some specialization.) This is applied to derive a short proof of a theorem proved by Gårding and Malgrange, in the constant coefficients case, concerning conditionally elliptic operators.

Let Ω be an open subset of R^n, $A = A(x, D)$ a strongly elliptic partial differential operator defined on Ω with analytic coefficients, of order $2m$.

THEOREM 1. *Let $u \in \mathfrak{D}'(\Omega)$. Suppose that for every relatively compact $\omega \subset \Omega$ and every $\phi \in C_0^\infty(\omega)$ we have $|\langle A^j u, \phi\rangle| \leqq C r^{2mj}(2mj)!$, where C and r are constants independent of j and r is independent of ϕ. Then u is analytic on Ω.*

PROOF. Consider $F = \sum_{j=0}^\infty ((-1)^{(m+1)j}/(2mj)!) t^{2mj} A^j u$. This series converges on $J = (-1/r, 1/r)$ to a continuous function with values in $\mathfrak{D}'(\omega)$. Hence $F \in \mathfrak{D}'(J \times \omega)$. Note that $(\partial^{2m}/\partial t^{2m}) F = (-1)^{m+1} A F$. But $(\partial^{2m}/\partial t^{2m}) + (-1)^m A$ is an elliptic operator on $J \times \omega$ with analytic coefficients, so F is analytic. Hence $u = F(0)$ is analytic.

THEOREM 2. *Suppose A_1 and A_2 are two differential operators of order $2m$ which commute, such that $A = A_1 + A_2$ is strongly elliptic of order $2m$. Let $u \in \mathfrak{D}'(\Omega)$ and suppose that $A_1 u$ is analytic while for every relatively compact open $\omega \subset \Omega$ and every $\phi \in C_0^\infty(\omega)$ we have $|\langle A_2^j u, \phi\rangle| \leqq C r^j (2mj)!$, where C and r are independent of j and r is independent of ϕ. Then u is analytic on Ω.*

PROOF. Since A_1 and A_2 commute we have

$$A^j u = A_2^j u + \sum_{k=0}^{j-1} \binom{j}{k+1} A_2^{j-k-1} A_1^k (A_1 u).$$

Received by the editors April 6, 1970.

AMS 1969 *subject classifications.* Primary 3543; Secondary 3524.

Key words and phrases. Elliptic operator, analytic function, conditionally elliptic operator.

[1] Research for this paper done while the author was supported by a National Science Foundation fellowship.

The hypotheses on $A_2^{\frac{1}{2}}u$ and A_1u guarantee that the hypotheses of Theorem 1 are satisfied for A and u, so u is analytic.

We apply this theorem to derive (a generalization of) a theorem of [1] concerning conditional ellipticity. Suppose R^n is decomposed as $R^n = R^k \times R^{n-k}$. If $x \in R^n$, write $x = (x', x'')$ with $x' \in R^k$, $x'' \in R^{n-k}$. A distribution $u \in \mathfrak{D}'(\Omega)$ is said to be analytic in x'' if each $x_0 \in \Omega$ has a neighborhood of the form $U \times V$ such that, on $U \times V$, $u = \sum_{\alpha \geqq 0} v_\alpha (x'' - x_0'')^\alpha$, $v_\alpha \in \mathfrak{D}'(U)$, such that for every $\psi \in C_0^\infty(U)$, $\sum_{\alpha \geqq 0} \langle v_\alpha, \psi \rangle (x'' - x_0'')^\alpha$ is a power series which converges absolutely and uniformly on V. It is well known, and easy to prove, that if u is analytic in x'', then for every relatively compact open $\omega \subset \Omega$ we have $\| D_{x''}^\alpha u \|_{H^s(\omega)} \leqq C^{|\alpha|+1} \alpha!$ for some C and s independent of α.

THEOREM 3. *Let $P = P(x', D)$ be a partial differential operator on Ω of order m with analytic coefficients which depend only on x'. Suppose $P(x', D) = P_0(x', D') + P_1(x', D)$ where $P_0(x', D')$ depends analytically on x' and is elliptic of order m in the arguments $D' = (D_1, \cdots, D_k)$, and where $P_1(x', D)$ is of order $\leqq m$, and of order $< m$ in D'. (We call such P a conditionally elliptic operator.) If $u \in \mathfrak{D}'(\Omega)$ is analytic in x'' and if Pu is analytic, then u is analytic.*

PROOF. Apply Theorem 2 to

$$A_1 = P^*P \quad \text{and} \quad A_2 = (-1)^m \alpha (\partial^{2m} / \partial x_{k+1}^{2m} + \cdots + \partial^{2m} / \partial x_n^{2m})$$

with $\alpha > 0$ sufficiently large that $A_1 + A_2$ is strongly elliptic, at least on any given relatively compact open subset of Ω.

BIBLIOGRAPHY

1. L. Gårding and B. Malgrange, *Opérateurs différentiels partiellement hypo-elliptiques et partiellement elliptiques*, Math. Scand. **9** (1961), 5–21. MR **23** #A3367.

2. T. Kotake and M. S. Narasimhan, *Regularity theorems for fractional powers of linear elliptic operators*, Bull. Soc. Math. France **90** (1962), 449–471. MR **26** #6819.

UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104