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THE CARDINALITY OF ULTRAPOWERS—AN EXAMPLE

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ABSTRACT. Assume the axiom of measurable cardinals. If D is an ω-incomplete uniform ultrafilter on I, and A is infinite, it is still not necessarily the case that A^I/D has the same cardinality as A^I.

There remain a number of unsolved problems about the cardinality of ultrapowers. We show here by means of simple examples that the situation is more complicated than it was hoped to be. Since we deal only with questions of cardinality, we do not distinguish between first-order structures and their underlying sets. In what follows, the cardinality of a set X is denoted by $|X|$.

The following questions about the cardinality of ultrapowers have remained open.

(a) Does there exist an infinite set I and a uniform ultrafilter D on I such that $\omega < |\omega^I/D| < 2^{|I|}$? (See [2], also [3] and [1].)

(b) Do there exist infinite sets I, A with $|A| \leqq |I|$ and a uniform ω-incomplete ultrafilter D on I such that $|A| = |A^I/D|$? ([3], [1].)

We show that if we accept the existence of a measurable cardinal, the answer to both these questions is positive.

Let E, F be ultrafilters on sets J, K respectively. It is known [4] that for any first-order structure A, $(A^J/E)^K/F \simeq A^{J\times K}/E\times F$, where $E\times F$ is the ultrafilter on $J\times K$ defined by:

$$X \in E \times F \quad \text{iff } \{k \in K \mid \{j \in J \mid \langle j, k\rangle \in X\} \in E\} \in F.$$

Assume that measurable cardinals exist. Let $|K|$ in particular be the first measurable cardinal, and let F be an ω-complete nonprincipal (necessarily uniform) ultrafilter on K. It is well known that if $|B| < |K|$, $B^K/F \simeq B$. Let $|J| = \omega$, and let E be a nonprincipal ultrafilter on J. Let $I = J \times K$, $D = E \times F$. It is easy to see that $|I| = |K|$, and that D is uniform and ω-incomplete.

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To answer question (a), let $|A| = \omega$. Then $A^I/D \simeq A^J/E$ and $|A^J/E| = 2^\omega$. Hence in particular we have certainly $\omega < |\omega^I/D| < 2^{|I|}$.

To answer question (b), let $|A| = 2^\omega$. Clearly $|I| > |A|$. Again $A^I/D \simeq A^J/E$. But $|A| = |A^J| = 2^\omega$, and hence $|A^I/D| = |A|$.

An ultrafilter D on I is said to be regular if there exists a subset X of D such that $|X| = |I|$ and the intersection of any infinite subset of X is empty. Keisler has shown that if A is infinite and D is regular, then $|A^I/D| = |A^I|$, and has asked whether every uniform ω-incomplete ultrafilter is regular [3]. It is clear from our examples for (a) or (b) that if we assume the axiom of measurable cardinals, not every countably incomplete uniform ultrafilter is regular.

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