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AN EXAMPLE OF HILTON AND ROITBERG

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ABSTRACT. In [3], P. J. Hilton and J. Roitberg illustrated with several examples the failure of cancellation for products in the homotopy category of finite CW complexes. We reconstruct here these examples from a different point of view.

1. **Introduction.** Each example of Hilton and Roitberg consisted of principal S^3-bundles E_α and E_β over S^n for which $E_\alpha \times S^3 \simeq E_\beta \times S^3$ and yet $E_\alpha \not\simeq E_\beta$. If B_{S^3} is the classifying space of the Lie group S^3 and if $\alpha \in \pi_{n-1}(S^3)$ and $\alpha_0 \in \pi_n(B_{S^3})$ correspond under the canonical isomorphism, then denote by $p_\alpha: E_\alpha \to S^n$ the principal S^3-bundle classified by $\alpha_0: S^n \to B_{S^3}$. They show that there is a supply of $\alpha, \beta \in \pi_{n-1}(S^3)$ with $\alpha \neq \pm\beta$, which guarantees that $E_\alpha \not\simeq E_\beta$, and with $p_\alpha \circ \beta_0 \simeq 0 \simeq p_\beta \circ \alpha_0$, which implies that the fibered product $E_{\alpha\beta}$ of the maps p_α and p_β satisfies $E_\alpha \times S^3 \simeq E_{\alpha\beta} \simeq E_\beta \times S^3$. The resulting homotopy equivalence $E_\alpha \times S^3 \to E_\beta \times S^3$, which is not made explicit in [3], cannot be of the form $f \times g$ for then $f: E_\alpha \to E_\beta$ would induce isomorphisms on homotopy and hence would be a homotopy equivalence; it must be twisted. We present here these examples from the point of view of the cellular structure of the spaces $E_\alpha \times S^3$ and $E_\beta \times S^3$ to indicate how the homotopy equivalence $E_\alpha \times S^3 \to E_\beta \times S^3$ can be generated by a twisted homotopy equivalence $S^3 \times S^3 \to S^3 \times S^3$.

Let $m_r: S^3 \times S^3 \to S^3$ $(r = 0, 1, \cdots, 11)$ be the twelve multiplications on S^3 as enumerated by M. Arkowitz and C. R. Curjel in [1]. For $\alpha: S^{n-1} \to S^3$ define the map

$$g_{\alpha,r} = \alpha \times 1 \circ m_r: S^{n-1} \times S^3 \to S^3 \times S^3 \to S^3$$

(observing the "Hilton-Wylie" convention of writing composition of maps) and the adjunction space $E_{\alpha,r} = S^3 \cup_{g_{\alpha,r}} B^n \times S^3$. These spaces are related to the principal S^3-bundles in that $E_{\alpha,0} = E_\alpha$ [3, Proposition 2.1]. We prove in §3 the following result.

THEOREM 1. *Let $\alpha, \beta: S^{n-1} \to S^3$ be used to construct $E_{\alpha,r}$ and $E_{\beta,s}$. If there exist integers n_{ij} $(i, j = 1, 2)$ such that*

(i) $\det(n_{ij}) = \pm 1$,

(ii) $n_{11}\alpha \simeq \beta$ *and* $n_{12}\alpha \simeq 0$,

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(iii) $n_{1j}^2(2s+1) \equiv n_{1j}(2r+1) \bmod 24$ $(j=1, 2)$,
and if,
(iv) *either* $s=0, 2, 3, 5, 6, 8, 9,$ *or* 11,
then there is a twisted homotopy equivalence $\{(\bar{n}_{ij})\}: S^3 \times S^3 \to S^3 \times S^3$ *which extends to a homotopy equivalence* $E_{\alpha,r} \times S^3 \to E_{\beta,s} \times S^3$.

For the remainder of this section we restrict our attention to $r=s$ from the list (iv). Using Theorem 1 we can reprove [**3**, Corollary 2.2] and [**3**, Theorem 2.5].

THEOREM 2. *Let* α *be of order* k, $k_0 = \gcd(k, 24)$, ι *prime to* k, $\iota \equiv 1$ mod k_0, $\beta = \iota\sigma$. *Then* $E_{\alpha,r} \times S^3 \simeq E_{\beta,r} \times S^3$.

PROOF. Since $\iota \equiv 1 \bmod k_0$, we can find $n_{11} \equiv \iota \bmod k$, $n_{11} \equiv 1 \bmod 24$. Then n_{11} is prime to k and to 24 and hence to $n_{12} = 24\,k$. Thus we have integers n_{ij} $(i, j=1, 2)$ with
(i) $\det(n_{ij}) = 1$,
(ii) $n_{11}\alpha \simeq \iota\alpha = \beta$ and $n_{12}\,\alpha = 0$, and
(iii) $n_{1j}(n_{1j}-1)(2r+1) \equiv 0 \bmod 24$ $(j=1, 2)$.
As in [**3**, Theorem 2.3] we can see that $E_{\alpha,r} \simeq E_{\beta,s}$ implies that $\alpha \simeq \pm\beta$ and so we have the immediate consequence of Theorem 2.

THEOREM 3. *Let* α *be of prime order* $p \neq 2, 3$, *let* ι *be prime to* p $\iota \not\equiv \pm 1 \bmod p$, $\beta = \iota\,\alpha$. *Then*

$$E_{\alpha,r} \times S^3 \simeq E_{\beta,r} \times S^3, \qquad E_{\alpha,r} \not\simeq E_{\beta,r}.$$

2. **The abstract situation.** We work in the category of k-spaces with base-point and base-point preserving maps, with composition of $f: A \to B$ and $g: B \to C$ written $f \circ g: A \to C$. The equivalence relation induced by homotopies which preserve base-points will be denoted by $\simeq$.

Given $g: X \times Y \to Z$ and the inclusion $c: X \to CX$ of X onto the base of its cone CX, the adjunction space $Z \cup_g CX \times Y$ described by

$$\begin{array}{ccc} X \times Y & \xrightarrow{c \times 1_Y} & CX \times Y \\ g \downarrow & & \downarrow \\ Z & \longrightarrow & Z \cup_g CX \times Y \end{array}$$

is Hausdorff and hence is a k-space [**4**, 2.6]. We may therefore consider the above diagram as a push-out in the category of k-spaces. Since there is an unrestricted exponential law in this category, each product functor $- \times W$ preserves push-outs and hence

PROPOSITION 4. *For any space* W, *the identity function*

$$1: Z \times W \cup_{g\times 1} CX \times Y \times W \to (Z \cup_g CX \times Y) \times W$$

is a homeomorphism.

PROPOSITION 5. *If* $h: X \to X'$, $k: Y \to Y'$, *and* $v: Z \to Z'$ *are homotopy equivalences, and if* $g: X \times Y \to Z$ *and* $g': X' \times Y' \to Z'$ *are maps such that* $g \circ v \simeq h \times k \circ g': X \times Y \to Z'$, *then there is a homotopy equivalence*

$$Z \cup_g CX \times Y \to Z' \cup_{g'} CX' \times Y'$$

extending $v: Z \to Z'$.

PROOF. When Y and Y' are singletons, then $Z \cup_g CX \times Y$ and $Z' \cup_{g'} CX' \times Y'$ are the mapping cones of g and g'. In this special case the above result is standard and its proof [**2**, p. 40] can easily be modified to cover the general case.

From now on let Y and Y' be connected cellular spaces with multiplications $m: Y \times Y \to Y$ and $m': Y' \times Y' \to Y'$ (i.e., the codiagonal maps $\nabla \simeq i \circ m: Y \vee Y \to Y \times Y \to Y$ and $\nabla \simeq i' \circ m': Y' \vee Y' \to Y' \times Y' \to Y'$). Then given $\alpha: X \to Y$ and $\beta: X \to Y'$ we form

$$g_\alpha = \alpha \times 1 \circ m: X \times Y \to Y \times Y \to Y,$$

$$g_\beta = \beta \times 1 \circ m': X \times Y' \to Y' \times Y' \to Y',$$

and the associated adjunction spaces $E_\alpha = Y \cup_{g_\alpha} CX \times Y$ and $E_\beta = Y' \cup_{g_\beta} CX \times Y'$. From the previous two propositions we have

COROLLARY 6. (i) $E_\alpha \times Y = Y \times Y \cup_{g_\alpha \times 1} CX \times Y \times Y$ *and* $E_\beta \times Y' = Y' \times Y' \cup_{g_\beta \times 1} CX \times Y' \times Y'$.

(ii) *If* $k: Y \times Y \to Y' \times Y'$ *is a homotopy equivalence for which*

$$\begin{tikzcd}
X \times Y \times Y \arrow[r, "1 \times k"] \arrow[d, "\alpha \times 1 \times 1"'] & X \times Y' \times Y' \arrow[d, "\beta \times 1 \times 1"] \\
Y \times Y \times Y \arrow[d, "m \times 1"'] & Y' \times Y' \times Y' \arrow[d, "m' \times 1"] \\
Y \times Y \arrow[r, "k"] & Y' \times Y'
\end{tikzcd}$$

is homotopy commutative, then k *extends to a homotopy equivalence* $E_\alpha \times Y \to E_\beta \times Y'$.

We now describe some special twisted homotopy equivalences $Y \times Y \to Y' \times Y'$. Let Z be a space with multiplication $n: Z \times Z \to Z$. Given four maps $k_{ij}: W \to Z$ $(i, j = 1, 2)$ we define $\{(k_{ij})\}: W \times W \to Z$

$\times Z$ to be the map with projections $\{(k_{ij})\} \circ p_j = p_1 \circ k_{1j} +_Z p_2 \circ k_{2j}$: $W \times W \to Z$ $(j=1, 2)$, where the sum $f +_Z g: A \to Z$ means $\Delta \circ f \times g \circ n$: $A \to Z$. For example, if $\delta_{ij}: Y \to Y$ is given by 0, 1_Y as $i \neq j$, $i=j$, then $\{(\delta_{ij})\} \simeq 1: Y \times Y \to Y \times Y$. We can consider the four maps $k_{ij}: W \to Z$ $(i, j=1, 2)$ as determining a 2×2 matrix (k_{ij}). We write $(k_{ij}) \simeq (h_{ij})$ if $k_{ij} \simeq h_{ij}$ $(i, j=1, 2)$, and so $(k_{ij}) \simeq (h_{ij})$ implies $\{(k_{ij})\} \simeq \{(h_{ij})\}: W \times W \to Z \times Z$. For $k_{ij}: Y \to Y'$ and $h_{ij}: Y' \to Y$ $(i, j=1, 2)$ we define matrix multiplication

$$(k_{ij})(h_{ij}) = \left(k_{i1} \circ h_{1j} \underset{Y}{+} k_{i2} \circ h_{2j}\right),$$

$$(h_{ij})(k_{ij}) = \left(h_{i1} \circ k_{1j} \underset{Y'}{+} h_{i2} \circ k_{2j}\right),$$

and we say (h_{ij}) and (k_{ij}) are inverses if these products $(k_{ij})(h_{ij}) \simeq (\delta_{ij})$ and $(h_{ij})(k_{ij}) \simeq (\delta'_{ij})$. We do not claim that then $\{(k_{ij})\} \circ \{(h_{ij})\} \simeq 1_{Y \times Y}$ and $\{(h_{ij})\} \circ \{(k_{ij})\} \simeq 1_{Y' \times Y'}$, but nevertheless we prove

PROPOSITION 7. *Given $k_{ij}: Y \to Y'$ $(i, j=1, 2)$, the map $\{(k_{ij})\}: Y \times Y \to Y' \times Y'$ is a homotopy equivalence if the matrix (k_{ij}) has an inverse.*

PROOF. For $g: S^n \to Y \times Y$, $(n \geqq 1)$,

$$\begin{aligned} g \circ \{(k_{ij})\} \circ p_j &= g \circ \left(p_1 \circ k_{1j} \underset{Y'}{+} p_2 \circ k_{2j}\right) \\ &= g \circ p_1 \circ k_{1j} \underset{Y'}{+} g \circ p_2 \circ k_{2j} \\ &\simeq g \circ p_1 \circ k_{1j} + g \circ p_2 \circ k_{2j} \qquad (j = 1, 2) \end{aligned}$$

where $+$ is the homotopy associative-commutative binary operation determined by the standard comultiplication on S^n. If (h_{ij}) is a matrix inverse for (k_{ij}) then

$$\{(k_{ij})\} \circ_\# \{(h_{ij})\}_\# = 1: \pi_n(Y \times Y) \to \pi_n(Y \times Y)$$

and

$$\{(h_{ij})\}_\# \circ \{(k_{ij})\}_\# = 1: \pi_n(Y' \times Y') \to \pi_n(Y' \times Y').$$

For example,

$$g \circ \{(k_{ij})\} \circ \{(h_{ij})\} \simeq g \quad \text{for } g: S^n \to Y \times Y \quad (n \geqq 1)$$

since

$$
\begin{aligned}
g \circ \{(k_{ij})\} \circ \{(h_{ij})\} \circ p_j &= g \circ \{(k_{ij})\} \circ \left(p_1 \circ h_{1j} \underset{Y}{+} p_2 \circ h_{2j}\right) \\
&= g \circ \{(k_{ij})\} \circ p_1 \circ h_{1j} \underset{Y}{+} g \circ \{(k_{ij})\} \circ p_2 \circ h_{2j} \\
&\simeq (g \circ p_1 \circ k_{11} + g \circ p_2 \circ k_{21}) \circ h_{1j} \\
&\quad + (g \circ p_1 \circ k_{12} + g \circ p_2 \circ k_{22}) \circ h_{2j} \\
&= (g \circ p_1 \circ k_{11} \circ h_{1j} + g \circ p_2 \circ k_{21} \circ h_{1j}) \\
&\quad + (g \circ p_1 \circ k_{12} \circ h_{2j} + g \circ p_2 \circ k_{22} \circ h_{2j}) \\
&\simeq (g \circ p_1 \circ k_{11} \circ h_{1j} + g \circ p_1 \circ k_{12} \circ h_{2j}) \\
&\quad + (g \circ p_2 \circ k_{21} \circ h_{1j} + g \circ p_2 \circ k_{22} \circ h_{2j}) \\
&\simeq \left(g \circ p_1 \circ k_{11} \circ h_{1j} \underset{Y}{+} g \circ p_1 \circ k_{12} \circ h_{2j}\right) \\
&\quad + \left(g \circ p_2 \circ k_{21} \circ h_{1j} \underset{Y}{+} g \circ p_2 \circ k_{22} \circ h_{2j}\right) \\
&= g \circ p_1 \circ \left(k_{11} \circ h_{1j} \underset{Y}{+} k_{12} \circ h_{2j}\right) \\
&\quad + \left(g \circ p_2 \circ (k_{21} \circ h_{1j} \underset{Y}{+} k_{22} \circ h_{2j}\right) \\
&\simeq g \circ \{(k_{ij})(h_{ij})\} \circ p_j \\
&\simeq g \circ \{(\delta_{ij})\} \circ p_j \simeq g \circ p_j
\end{aligned}
$$

for $j=1, 2$.

Thus $\{(k_{ij})\}: Y\times Y \to Y' \times Y'$, as a weak homotopy equivalence between connected cellular spaces, is a homotopy equivalence.

THEOREM 8. *Let $\alpha: X \to Y$ and $\beta: X \to Y'$ be used to construct E_α and E_β as prior to Corollary* 6. *If there exist four maps k_{ij}: $Y \to Y'$ $(i, j=1, 2)$ such that*

(i) *the matrix (k_{ij}) is invertible,*

(ii) *$\alpha \circ k_{11} \simeq \beta: X \to Y'$ and $\alpha \circ k_{12} \simeq 0: X \to Y'$,*

(iii) *k_{1j}: $Y \to Y'$ is an H-map for $j=1, 2$, and if*

(iv) *the multiplication m': $Y' \times Y' \to Y'$ is homotopy associative,*

then the map $\{(k_{ij})\}: Y\times Y \to Y' \times Y'$ is a homotopy equivalence which extends to a homotopy equivalence $E_\alpha \times Y \to E_\beta \times Y'$

PROOF. Condition (i) and the previous proposition show that $\{(k_{ij})\}$ is a homotopy equivalence. We use conditions (ii), (iii), and (iv) to show that

$$\alpha \times 1 \times 1 \circ m \times 1 \circ \{(k_{ij})\} \simeq 1 \times \{(k_{ij})\} \circ \beta \times 1 \times 1 \circ m' \times 1,$$

so that Corollary 6 is applicable:

$$
\begin{aligned}
&\alpha \times 1 \times 1 \circ m \times 1 \circ \{(k_{ij})\} \circ p_j \\
&\qquad\qquad \simeq \alpha \times 1 \times 1 \circ m \times 1 \circ \left(p_1 \circ k_{1j} \underset{Y'}{+} p_2 \circ k_{2j}\right) \\
&\qquad\qquad \simeq \alpha \times 1 \times 1 \circ m \times 1 \circ k_{1j} \times k_{2j} \circ m' \\
&\qquad\qquad \simeq \alpha \times 1 \times 1 \circ k_{1j} \times k_{1j} \times k_{2j} \circ m' \times 1 \circ m' \\
&\qquad\qquad \simeq (\alpha \circ k_{1j}) \times k_{1j} \times k_{2j} \circ 1 \times m' \circ m' \qquad (j = 1, 2)
\end{aligned}
$$

while

$$
\begin{aligned}
&1 \times \{(k_{ij})\} \circ \beta \times 1 \times 1 \circ m' \times 1 \circ p_1 \\
&\qquad\qquad = 1 \times (\{(k_{ij})\} \circ p_1) \circ \beta \times 1 \circ m' \\
&\qquad\qquad \simeq 1 \times (k_{11} \times k_{21} \circ m') \circ \beta \times 1 \circ m' \\
&\qquad\qquad = 1 \times k_{11} \times k_{21} \circ 1 \times m' \circ \beta \times 1 \circ m' \\
&\qquad\qquad = \beta \times k_{11} \times k_{21} \circ 1 \times m' \circ m' \\
&\qquad\qquad \simeq (\alpha \circ k_{11}) \times k_{11} \times k_{21} \circ 1 \times m' \circ m'
\end{aligned}
$$

and

$$
\begin{aligned}
1 \times \{(k_{ij})\} \circ \beta \times 1 \times 1 \circ m' \times 1 \circ p_2 &= p_{Y \times Y} \circ \{(k_{ij})\} \circ p_2 \\
&\simeq p_{Y \times Y} \circ k_{12} \times k_{22} \circ m' \\
&\simeq 0 \times k_{12} \times k_{22} \circ 1 \times m' \circ m' \\
&\simeq (\alpha \circ k_{12}) \times k_{12} \times k_{22} \circ 1 \times m' \circ m',
\end{aligned}
$$

where $p_{Y \times Y}: X \times Y \times Y \to Y \times Y$ is projection on the last two factors.

3. **A concrete case.** Let $X = S^{n-1}$, let Y be the three sphere S^3 with multiplication m_r, and let Y' be the three sphere S^3 with multiplication m_s. We claim that Theorem 1 is merely a rewording of Theorem 8 and we present the following facts in justification:

(i) For each integer n, let $[n]: S^3 \to S^3$ be a map of degree n. Since $[n+m] \simeq [n] + [m]$, where $+ = +_Y$, $+_{Y'}$, and $[n] \circ [m] \simeq [nm]$, then

$$
\begin{aligned}
([n_{ij}])([m_{ij}]) &= ([n_{i1}] \circ [m_{1j}] + [n_{i2}] \circ [m_{2j}]) \\
&\simeq ([n_{i1} m_{1j} + n_{i2} m_{2j}]).
\end{aligned}
$$

We conclude that the matrix of maps $([n_{ij}])$ is invertible iff the matrix of integers (n_{ij}) is invertible, or equivalently, iff $\det (n_{ij}) = \pm 1$.

(ii) For $\alpha: S^{n-1} \to S^3$ and integer n, we have $\alpha \circ [n] \simeq n\alpha$.

(iii) The map $[n]: S^3, m_r \to S^3, m_s$ is an H-map iff $n^2(2s+1) \equiv n(2r+1) \bmod 24$ [1, Theorem A].

(iv) The multiplication $m_s: S^3 \times S^3 \to S^3$ is homotopy associative iff $s = 0, 2, 3, 5, 6, 8, 9$, or 11 [1, Theorem B and Remark 1].

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