

A TRANSITIVE MEDIAL SUBSPACE LATTICE

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ABSTRACT. We give an example of a transitive medial lattice of subspaces of Hilbert space which has four nontrivial elements.

Let \mathcal{H} be a separable infinite-dimensional Hilbert space (real or complex). A *subspace* of \mathcal{H} is a closed linear manifold. If A is a bounded linear operator on \mathcal{H} , then the collection of all subspaces invariant under A is a lattice. There has been some study of the lattices that can occur as invariant subspace lattices (cf. [2] and references given there).

Recently Halmos [1] initiated the study of the lattices that can occur as the invariant subspace lattices of collections of operators.

A subspace lattice is *transitive* if the only bounded operators which leave invariant all the subspaces in the lattice are the multiples of the identity operator. The basic problem of characterizing transitive lattices seems to be difficult. A lattice with 0 and 1 is said to be *medial* if every pair of distinct nontrivial (i.e. neither 0 nor 1) elements are complements of each other. Halmos [1] has given an example of a transitive medial subspace lattice with 5 nontrivial elements, and has shown that every transitive medial lattice of subspaces of a finite-dimensional space of (complex) dimension greater than 2 has at least 5 nontrivial elements.

In this note we give an example of a transitive medial lattice of subspaces of separable Hilbert space which has 4 nontrivial elements. We think that the number 4 is minimal, but we have been unable to prove it.

To construct the example let \mathcal{K} be a Hilbert space with orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. Let $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and let \mathfrak{M}_{x0} , \mathfrak{M}_{0x} and \mathfrak{M}_{xx} denote the subspaces $\mathcal{K} \oplus 0$, $0 \oplus \mathcal{K}$ and $\{(x, x) : x \in \mathcal{K}\}$ of \mathcal{H} . Let $\alpha_n = 1$ for $n \leq 0$ and $\alpha_n = \exp((-1)^n n!)$ for $n > 0$. Let T be the operator defined by

$$T\left(\sum_{n=-\infty}^{\infty} y_n e_n\right) = \left(\sum_{n=-\infty}^{\infty} y_n \alpha_n e_{n+1}\right)$$

on the domain $\mathfrak{D} = \left\{ \sum_{n=-\infty}^{\infty} y_n e_n : \sum_{n=-\infty}^{\infty} |y_n \alpha_n|^2 < \infty \right\}$. Then it is

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easily verified that T is a closed operator, i.e., that the subspace $\mathfrak{N}_T = \{(x, Tx); x \in \mathfrak{D}\}$ is a closed subspace of \mathfrak{K} .

THEOREM. *The set $\{\{0\}, \mathfrak{K}, \mathfrak{N}_{x_0}, \mathfrak{N}_{0x}, \mathfrak{N}_{xx}, \mathfrak{N}_T\}$ is a transitive medial lattice.*

PROOF. We first show that the set is a medial lattice. For this we shall need some facts about T :

- (i) The closure of the range of T is \mathfrak{K} , since $e_n \in \text{range } T$ for all n .
- (ii) The closure of the range of $(T-1)$ is \mathfrak{K} , since if $x = \sum x_n e_n$ is orthogonal to $(T-1)e_n$ for all n , then $\alpha_n x_{n+1} - x_n = 0$ for all n . Therefore, since $\alpha_n = 1$ for $n \leq 0$, $x_n = x_0$ for all negative n and thus $x_0 = 0$. Since every α_n is positive, it follows that $x_n = 0$ for all n .
- (iii) $\text{Null } T = \{0\}$, since $\alpha_n \neq 0$ for all n .
- (iv) $\text{Null } (T-1) = \{0\}$, for if $(T-1)\sum x_n e_n = 0$, then $x_{n+1} = \alpha_n x_n$, and, as in (ii), we get $x_n = 0$ for all n .

We must show that any two nontrivial elements of the lattice are complementary.

- (a) $\mathfrak{N}_{x_0} \cap \mathfrak{N}_T = \{0\}$ by (iii), and $\mathfrak{N}_{x_0} \vee \mathfrak{N}_T = \mathfrak{K}$ by (i).
- (b) $\mathfrak{N}_{xx} \cap \mathfrak{N}_T = \{0\}$ by (iv), and $\mathfrak{N}_{xx} \vee \mathfrak{N}_T = \mathfrak{K}$ by (ii).

The other pairs are trivially seen to be complementary, and thus the set is a medial lattice.

We now show that the lattice is transitive. Obviously every operator which leaves \mathfrak{N}_{x_0} , \mathfrak{N}_{0x} and \mathfrak{N}_{xx} invariant can be written $A \oplus A$ where A is an operator on \mathfrak{K} . We must show that A is a multiple of the identity. It follows from the invariance of \mathfrak{N}_T under $A \oplus A$ that \mathfrak{D} is invariant under A and that $AT = TA$ on \mathfrak{D} .

Note that for every $x \in \mathfrak{D}$, $(Tx, e_n) = \alpha_{n-1}(x, e_{n-1})$. Since $(ATe_{m-1}, e_n) = (TAe_{m-1}, e_n)$ for all m and n , it follows that $\alpha_{m-1}(Ae_m, e_n) = \alpha_{n-1}(Ae_{m-1}, e_{n-1})$. Thus (Ae_n, e_n) is a constant independent of n . If A were not a multiple of the identity, there would exist distinct integers m and n such that $(Ae_m, e_n) \neq 0$. But the above computation implies that

$$(Ae_{m+k}, e_{n+k}) = \frac{\alpha_n \alpha_{n+1} \cdots \alpha_{n+k-1}}{\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}} (Ae_m, e_n) \quad \text{for } k > 0.$$

The proof will be completed if we show that (Ae_{m+k}, e_{n+k}) is not bounded as $k \rightarrow \infty$, because this would contradict the fact that A is bounded.

The fraction on the right-hand side of the above equation can be rewritten as

$$c_1 \alpha_{m+k} \alpha_{m+k-1} \cdots \alpha_{m+1} \quad \text{OR} \quad c_2 (\alpha_{n+k} \alpha_{n+k+1} \cdots \alpha_{m+k-1})^{-1}$$

according as $n > m$ or $m > n$, where c_1 and c_2 are nonzero numbers not depending on k . In both cases the fraction is unbounded: just observe that if r is a fixed integer and if

$$P_j = \alpha_j \alpha_{j+1} \cdots \alpha_{j+r},$$

then $\limsup_{j \rightarrow \infty} P_j = \infty$ and $\liminf_{j \rightarrow \infty} P_j = 0$.

REFERENCES

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