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COMPLETION OF NORMS FOR $C(X, Q)$

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ABSTRACT. Let $C(X, Q)$ denote the algebra of all continuous quaternion-valued functions vanishing at infinity on a locally compact Hausdorff space X. Under the natural norm (the sup norm) and under the spectral radius norm, $r(f)$, which is equivalent to the sup norm, $C(X, Q)$ is a Banach algebra. Let $\delta(f)$ be any multiplicative norm for $C(X, Q)$; i.e., one under which it is a normed algebra. It is shown that $\delta(f)$, whether or not it is complete, majorizes the natural norm and $r(f)$. Under certain conditions on the radical of the completion of $\delta(f)$, $\delta(f)$ is equivalent to the natural norm and $r(f)$.

Let B be a real Banach algebra. Let $H(B, Q)$ denote the set of algebraic homomorphisms of B into the quaternions, with the weak topology introduced in H. Denote the zero homomorphism by π_0 and the image of $x\in B$ under $\pi\in H$ by $x(\pi)$. B is called normal if for every two closed disjoint sets F, C in H, with $\pi_0\notin C$, there exists an $x\in B$ such that $x(\pi)=0$ for all $\pi\in F$ and $x(\pi)=1$ for all $\pi\in C$. B is called regular if for every closed set F in H and for every $\pi'\in H-F$, $\pi'\neq\pi_0$, there exists an $x\in B$ such that $x(\pi)=0$ for all $\pi\in F$ and $x(\pi')\neq 0$.

Let $s_B(x)$ denote the spectrum of x in B and let $r_B(x)$ denote the spectral radius of x in B.

An algebra is said to satisfy *Property* M if each of its modular maximal right ideals is two-sided. An algebra is said to be strictly *semisimple* [5] if the intersection of its modular maximal right ideals which are two-sided is zero.

LEMMA. *Let B be a Banach algebra which has an identity and satisfies Property* M. *Then B is normal if and only if B is regular.*

THEOREM 1. *Let B be a strictly semisimple normal Banach algebra that is algebraically embedded in a second Banach algebra B'. If B satisfies Property* M, *then every member of $H(B, Q)$ can be extended to a member of $H(B', Q)$ and for every $x\in B$, $s_B(x)=s_{B'}(x)$ and $r_B(x)=r_{B'}(x)$.*

THEOREM 2. *Let B be a strictly semisimple normal Banach algebra that satisfies Property* M. *Let $\delta(x)$ be any multiplicative norm (not*

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necessarily complete) for B. Let B' denote the completion under the norm $\delta(x)$. Then for every $x \in B$, $s_B(x) = s_{B'}(x)$, $r_B(x) = r_{B'}(x)$, and $r_B(x) \leqq \delta(x)$.

Theorems 1 and 2 can be established by an adaptation of the proofs in the commutative case [**4**], [**10**]. Note that $r_{B'}(x) \leqq \delta(x)$ and since $r_B(x) = r_{B'}(x)$, $r_B(x) \leqq \delta(x)$.

COROLLARY. *For $f \in C(X, Q)$, $r(f) \leqq \delta(f)$ and* $\sup|f| \leqq \delta(f)$.

Cf. Olubummo [**7**] for another proof of the minimal norm property of the sup norm which uses a result of Rickart [**9**], [**10**]. Our corollary is an analogue of Kaplansky's result concerning the minimal character of the natural norm for $C(X)$, with the quaternions replaced by the reals or complexes [**3**, Theorem 6.2]. Kaplansky goes on to show (Theorem 6.3) that any multiplicative norm for $C(X)$ in which the completion is semisimple is equivalent to the natural norm. Our remaining theorems are noncommutative analogues.

THEOREM 3. *$\delta(f)$ is equivalent to the natural norm if the completion of $C(X, Q)$ under $\delta(f)$ has any one of the following properties:*

(a) *It is strictly semisimple.*

(b) *It is strongly semisimple.*

(c) *Zero is the only topologically nilpotent element.*

PROOF. (a) A strictly semisimple Banach algebra has the property that any (algebraic) homomorphism into it is continuous [**6**].

(b) Yood [**12**, Theorem 3.5] has shown that any homomorphism onto a dense subset of a strongly semisimple Banach algebra is continuous.

(c)[1] Let B' denote the completion of $C(X, Q)$ under $\delta(f)$. If zero is the only topologically nilpotent element of B', then any subalgebra of it is semisimple [**8**]. Johnson [**2**] has established the uniqueness of norm property for a semisimple Banach algebra. But a homomorphism onto a semisimple Banach algebra with a unique norm is continuous [**8**].

An appeal to the preceding corollary completes the proof since $\delta(f) \leqq r(f)$ and by the corollary $r(f) \leqq \delta(f)$ and $\sup|f| \leqq \delta(f)$.

THEOREM 4. *$\delta(f)$ is equivalent to the natural norm if the following conditions hold:*

(a) *The completion is semisimple.*

(b) *$r(f_n) \to r(f)$ when $\delta(f_n - f) \to 0$ as $n \to \infty$.*

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[1] This proof for part (c) was suggested by Carolyn Podest Klimek.

PROOF. Let B' denote the completion. Under the hypotheses it can be shown that

$$\text{(c)} \qquad \begin{aligned} r_{B'}(f+g) &\leqq r_{B'}(f) + r_{B'}(g), \\ r_{B'}(fg) &\leqq r_{B'}(f) r_{B'}(g). \end{aligned}$$

This can be used to show that the set N of topologically nilpotent elements in B' is a two-sided ideal in B'. [Suppose $f \in N$. Then for any $g \in B'$ and any real scalar α, $\alpha g = g_1 \in B'$ so that, from (c), $r_{B'}(fg_1) \leqq r_{B'}(f) r_{B'}(g_1) = 0$ since $r_{B'}(f) = 0$. Therefore $fg_1 = f\alpha g = \alpha fg \in N$. If $f, g \in N$, then, from (c), $r_{B'}(f+g) \leqq r_{B'}(f) + r_{B'}(g) = 0$ so $f+g \in N$.]

Thus N is a left and similarly a right ideal in B'. Since it is a topologically nilpotent ideal, it is contained in the Jacobson radical. By hypothesis B' is semisimple and therefore $N = (0)$. Now one can appeal to the previous theorem. Or one can show that $r_{B'}(f)$ on B' satisfies the condition that there exists a function on B' which is a nonnegative subadditive function that vanishes only for 0 and is majorized by $r_{B'}(f)$; this ensures continuity of homomorphisms into B' (Yood, [11]).

It would be desirable to delete condition (b) in Theorem 4.

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